KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLII

No. 6

President: J. VAN DER HOEVE
Secretary: M. W. WOERDEMAN

CONTENTS

JONKER, G. H., H. R. KRUYT and L. S. ORNSTEIN: "Formation and properties of the silver bromide sol", p. 454.

CORPUT, J. G. VAN DER: "Une inégalité relative aux sommes de WEYL", p. 461.

CORPUT, J. G. VAN DER: "Sur la méthode des points décisifs", p. 468.

CORPUT, J. G. VAN DER, et CH. PISOT: "Sur la discrépance modulo un". (Première communication), p. 476.

VISSER, CORNELIS: "On a theorem in the geometry of numbers and a property of mass distributions in *n*-dimensional space". (Communicated by Prof. J. G. VAN DER CORPUT), p. 487.

MONNA, A. F.: "On the DIRICHLET Problem and the Method of Sweeping-out". (Communicated by Prof. W. VAN DER WOUDE), p. 491.

DOLS, M. J. L., B. C. P. JANSEN, G. J. SIZOO and G. J. VAN DER MAAS: "Studies on phosphorus metabolism in normal and rachitic chickens with a radioactive phosphorus isotope". III. The distribution of the injected phosphorus in the organs, especially in the different parts of the leg bones. (Communicated by Prof. G. VAN RIJNBERK). (With one plate), p. 499.

DEMUTH, FRITZ: "Ueber die Beziehungen des Energiestoffwechsels zu Wachstum und Differenzierung". II. Die Wirkung von Sauerstoff auf Wachstum und Entwicklung von Hühnerembryonen. (Communicated by Prof. J. BOEKE). (With two plates and one table), p. 506.

BOK, S. T.: "Cephalization and the boundary values of the brain- and body sizes in mammals". (Communicated by Prof. M. W. WOERDEMAN), p. 515.

MEADER, R. G.: "On the Time and Place of Experimental Myelin Degeneration in the Optic Pathways". (Communicated by Prof. B. BROUWER). (With one plate), p. 526.

Erratum, p. 533.

Physics. — *Formation and properties of the silver bromide sol.* By G. H. JONKER, H. R. KRUYT and L. S. ORNSTEIN.

(Communicated at the meeting of May 20, 1939.)

1. *Introduction.*

If we mix strongly diluted solutions (10^{-4} — 10^{-6} N) of $AgNO_3$ and KBr, an AgBr sol is formed which originally is transparent but in the course of a few hours becomes gradually more turbid, until a certain degree of turbidity is reached, dependent upon various factors.

According to JABLCZYNSKI [1]), the increasing turbidity is due to coarsening as the result of recrystallization, while SCHNELLER [2]) mainly considers the process as a flocculation phenomenon.

We have once more examined in detail which changes take place in the sol, by measuring the conductivity, extinction, Br-ion concentration and electrophoretic velocity.

2. *Conductivity.*

The measurements of the conductivity were carried out with an apparatus built according to the directions of JONES [3]). 25 cc of $2 . 10^{-4}$ N $AgNO_3$ solution and 25 cc of $2,2 . 10^{-4}$ N KBr solution were during 20 minutes heated in a thermostat of 25°,00 C. These solutions were quickly mixed and poured into 3 flocculation glasses hung in the thermostat. In the first glass the conductivity cell was placed during 2 minutes in order to be brought into adsorption equilibrium. Subsequently the cell was placed alternately in the 2nd and 3rd glass and each time the resistance was determined, till after a few hours it remained constant.

Immediately after mixing there are present in the solution Ag^+-, NO_3^--, K^+- and Br^--ions. The conductivity of this mixture can be calculated. It is about $30 . 10^{-6}$ Ohm^{-1} cm^{-1}. The greater part of the Ag^+- and Br^--ions disappear from the solution. Consequently it is to be expected that the conductivity will be lowered to about one half.

The first measurement was made 3 to 5 minutes after mixing. From the fact that the conductivity then already had reached half the calculated initial value and even began to increase it follows that within this time all the AgBr had crystallized and that consequently the sols further contained a constant quantity of AgBr.

In fig. 1 the course of the conductivity has been drawn. The drawn part of the curve has been really measured. The further course is not easily interpreted, since the two parts of the conductivity, of electrolyte and of colloid, may change separately.

---

[1]) C. JABLCZYNSKI, Bull. Soc. Chim. France [IV] **33**, 1392 (1923).

[2]) H. SCHNELLER, Koll. Zeit. **71**, 180 (1935).

[3]) G. JONES, J. Am. Chem. Soc. **50**, 1049 (1928).

3. *Extinction.*

The extinction measurements of sols with 10^{-4} mol AgBr per liter were performed with an extinction meter with vacuum thermo-elements, according to Moll.

For the more strongly diluted sols we took a photo-electric selenium cell.

For all extinction measurements white light was used. 100 cc of $AgNO_3$ solution were mixed with 100 cc of KBr solution and poured into a cuvette with a thickness of 30 mm. The extinction was calculated in percentages of the light transmitted by the water.

There were the following influences upon the extinction of the sol:

a. Impurities caused by dust. Dust particles serve as nuclei for the crystallization and consequently have a large influence upon the number of the first formed crystals.

This influence is clearly apparent from the extinction curves in fig. 2. For sol *a* the solutions were filtered through a g_3 glass filter, for sol *b* through the denser g_4 filter and for sol *c* through a membrane filter.

Owing to the dust, the extinction curves could not be well reproduced. We tried to correct this by beforehand introducing a certain large amount of particles into the solution, viz. by adding some cc of a similar sol to the KBr solution. The influence of this on the extinction curves was very large, as may be seen in fig. 3. However, this was not the right way to examine the sol formation, for an important part, the formation of nuclei of their own accord during crystallization was eliminated here. For the following measurements, therefore, the solutions were made as pure as possible.

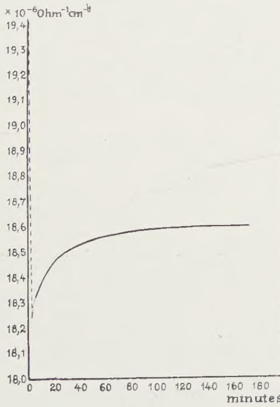


Fig. 1

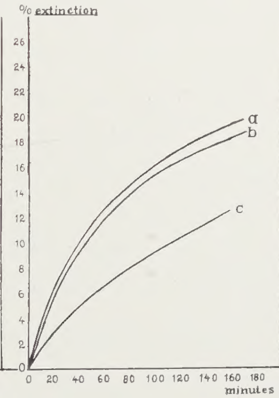


Fig. 2

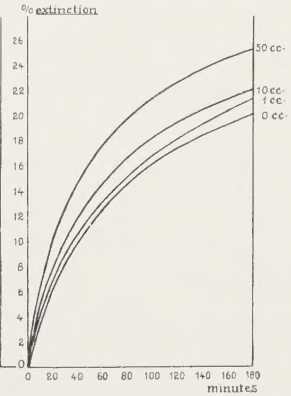


Fig. 3

b. The temperature had a large influence upon the rate of increase and on the final value of the extinction. The constant temperature, which consequently was necessary, was obtained by placing a glass tube in the cuvette, through which water was pumped from a thermostat. The measurements were carried out at 25°.

c. The extra concentration of KBr was the factor which proved to have the largest influence. As the result of this, for the sols with 10^{-4} mol AgBr per liter the extinction curve of the sol with equivalent quantities of $AgNO_3$ and KBr could not be determined accurately, owing to small errors made during pipetting.

Owing to greater excess of KBr the extinction curve was constantly lower. This has been reproduced for the sols with 10^{-4} mol AgBr per liter in fig. 4 and for the sols with $2 \,.\, 10^{-5}$ mol AgBr per liter in fig. 5.

d. The sol concentration influenced the shape of the curve. While for the sols with 10^{-4} mol AgBr per liter it has a monotonous course, for those with $2 \,.\, 10^{-5}$ and $2 \,.\, 10^{-6}$ mol AgBr per liter the S-form occurs. See figs. 4 and 5.

From the increase of the extinction it follows that either the particles grow constantly larger or the number of particles grows smaller.

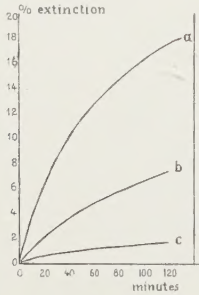


Fig. 4

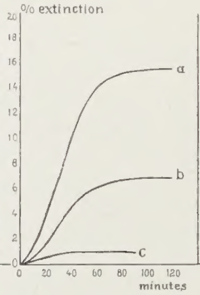


Fig. 5

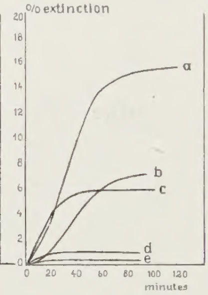


Fig. 6

Fig. 4. *a.* 10^{-6} mol KBr/liter
b. 10^{-5} ,, ,, ,,
c. 10^{-3} ,, ,, ,,

Fig. 5. *a.* $2 \,.\, 10^{-6}$ mol KBr/liter
b. $2 \,.\, 10^{-5}$,, ,, ,,
c. $2 \,.\, 10^{-4}$,, ,, ,,

Fig. 6. *a.* 0 mg
b. 0.5 ,,
c. 3 ,,
d. 50 ,,
e. 500 ,,

According to Rayleigh, the dispersion of the light by a particle is proportional to the square of the volume *a*, e.g. ka^2. The dispersion by *n* particles is then $k\,n\,a^2$. If the total volume of the solid is *v*, then is $n = \frac{v}{a}$, so that the dispersion is $k\,v\,a$. According to the conductivity test *v* is constant, so that he extinction is proportional to the average size of the particles. When the particles are no longer small with regard to the wavelength of the light and not globular, this relation is no longer applicable, but yet the extinction remains a criterion for the size of the particles.

4. *Gelatin tests.*

From the measurements mentioned sub 3 it does not yet follow in what way the particles grow larger.

By means of experiments with gelatin we could make a decision between recrystallization and flocculation. Recrystallization, i.e. the dissolving of small particles and the developing of larger ones, is a diffusion process. This cannot be retarded by gelatin. Flocculation, however, may be completely prevented by gelatin.

From our experiments it appeared that even small quantities of gelatin (0.5 to 3 mg per 100 cc of sol) strongly lower the extinction curve and that higher concentrations (500 mg) almost completely prevent turbidity (fig. 6). This proves that the coarsening consists of a flocculation, although to a small extent recrystallization may take place as well.

It is now evident why the extra concentration of KBr so strongly influences the rate of flocculation and its final value. The KBr, namely, regulates the stability of the particles.

It seems that for the explanation of this flocculation the theory of KRUYT and DE HAAN [4]) on the retarded flocculation may be applied. Immediately after formation of the sol the total surface of the particles is so large that the added quantity of KBr is not sufficient for the formation of a complete double layer. Moreover, this double layer may be distributed irregularly owing to the rapid formation, so that the particles show charged and uncharged spots. Upon collision of two particles with uncharged spots, adhesion may easily take place. Thus floccules are formed, which are gradually more completely surrounded by charged spots, so that the flocculation proceeds constantly more slowly and finally comes to a stop. Simultaneously the electrophoretic velocity must increase.

According to the researches by KRUYT and VERWEY [5]), the surface decreases in size as the result of internal recrystallization. Owing to this, a peptizing electrolyte is liberated, involving greater density of charge and consequently greater stability.

5. *Br^--concentration measurements.*

In order to find out whether during the extinction of the AgBr sol really a peptizing electrolyte was liberated, Br^--ion concentrations were determined at different times.

For this purpose was measured the potential difference between an AgBr electrode in the sol and a calomel electrode saturated with KCl, which by means of a liquid junction ($1{,}7\,N\,KNO_3 + 0{,}3\,N\,NaNO_3$) was connected with the sol. The potential difference was measured with a lamp voltmeter. The Br^--concentration was found by interpolation between the E.M.F.'s belonging to standard solutions with 10^{-4}, 10^{-5} and 10^{-6} g. equiv. Br^- per liter.

For these measurements sols were prepared with 10^{-4} g. mol AgBr/liter and different quantities of KBr.

---

[4]) H. R. KRUYT, These Proceedings **32**, 857 (1929); H. R. KRUYT and E. F. DE HAAN, Koll. Zeit. **51**, 61 (1930).

[5]) E. J. W. VERWEY and H. R. KRUYT, Z. physik. Chem. A **167**, 137 (1933).

In most cases the Br— conc. was lowered during some minutes, then remained constant for a long time and finally began to rise.

Sol + 0,5 . 10^{-4} g mol KBr/l.		Sol + 0,1 . 10^{-4} g mol KBr/l.		Equivalent sol	
Time	Br^- conc.	Time	Br^- conc.	Time	Br^- conc.
1 min	3,8 . 10^{-5} mol/l.	2 min	0,95 . 10^{-6} mol/l.	1 min	0,21 . 10^{-6} mol/l.
26 „	3,8 . 10^{-5} „	5 „	0,78 . 10^{-6} „	5 „	0,26 „
2½ hrs	4,0 . 10^{-5} „	6 hrs	0,78 . 10^{-6} „	30 „	0,34 „
21 „	4,6 . 10^{-5} „	23 „	1,40 . 10^{-6} „	42 „	0,34 „
		29 „	2,40 . 10^{-6} „	6 hrs	0,92 „
				23 „	3,40 „

The course of the Br— conc. is remarkable in the equivalent sol. The solubility of AgBr in water is at 18° C. 0,7 . 10^{-6} g. mol/1. The Br— conc. of the equivalent sol, however, is immediately after preparation 0,21 . 10^{-6}. Consequently Br— has been adsorbed originally. After 6 hours the concentration of the bromine ions had increased to 0,92 . 10^{-6} and after 20 hours even to 3,40 . 10^{-6}, i.e. far above the value corresponding to the solubility. The sol consequently showed a reversed adsorption; now Ag^+-ions were adsorbed.

6. *Electrophoretic measurements.*

It became now necessary to examine the change in stability of the sol. For this purpose was used the apparatus for electrophoretic measurements according to VAN GILS [6]). Sols with 2 . 10^{-5} g. mol AgBr were immediately fit for measuring.

The first measurements were made on a sol with 0,2 . 10^{-5} g. mol KBr per liter extra. The sol showed a negative electrophoretic velocity, which during the extinction increased. Also after the period required for reaching the final value of the turbidity the increase in velocity continued (Fig. 7*a*). Sols with greater excess of KBr showed the same. For the equivalent sol, considering the course of the Br— conc., it was to be expected that it would change from negative to positive. However, this was not the case. Like the sols with excess of KBr, it became increasingly negative (Fig. 7*b*).

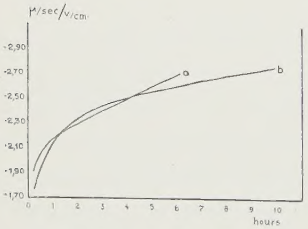


Fig. 7

Similarly a sol with 0,4 . 10^{-5} mol/liter excess $AgNO_3$, which according to § 5 will adsorb Ag^+ still more strongly, was immediately after formation negative and became more strongly negative. A sol with greater excess, 2 . 10^{-5} g. mol $AgNO_3$ per liter, was immediately after mixing positive. This sol caused a surprise. The electrophoretic velocity decreased rapidly and after $1\frac{1}{2}$ hours the sol had become negative. This was likewise the case with sols with 8 . 10^{-5} and 18 . 10^{-5} g. mol $AgNO_3$ per liter. With these sols the reversal set in after 10 and 20 hours respectively.

---

[6]) G. E. VAN GILS and H. R. KRUYT, Koll. Beihefte **45**, 60 (1936).

Consequently, in spite of the Ag^+ adsorption, all examined sols have turned negative (Fig. 8).

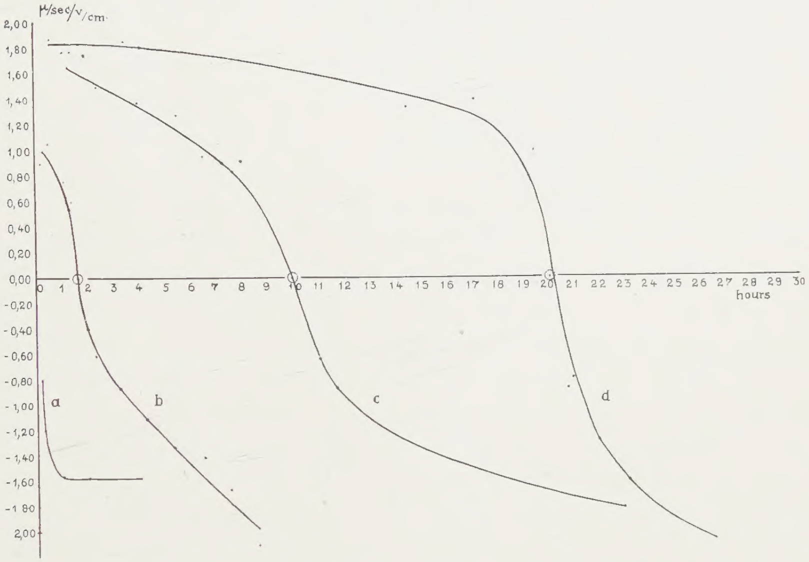


Fig. 8.

The same phenomenon was found in a different way by JULIEN [7]. JULIEN measured flow potentials on AgBr capillaries. It proved impossible to give the wall of the capillary a positive charge with $AgNO_3$ solutions. Only if on the wall fresh AgBr was precipitated, he succeeded but when the precipitate was some hours old it could no longer be done. JULIEN expected that a smooth AgBr surface like most substances in water would acquire a negative charge.

This may be a good explanation for the behaviour of the AgBr sols. It has to be assumed then that on the crystal surfaces, which during extinction grow constantly larger, a new double layer is formed by the side of the normal one on the edges and angles.

This new double layer has such a large influence that originally positive sols turn negative.

7. *Influence of electrolytes.*

In order to demonstrate the existence of these two double layers by the side of each other, we also examined the influence of various neutral electrolytes on the electrophoretic velocity of a 12 hours old, so negative sol with $2 \, . \, 10^{-5}$ g. mol $AgNO_3$ per liter in excess. The negative ions influence the normal double layer, the positive ions the new double layer of the crystal surfaces.

---

[7] P. F. J. A. JULIEN, Thesis. Utrecht 1933.

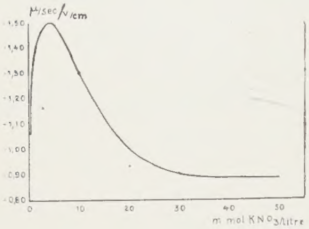


Fig. 9

1°. KNO_3. Small quantities of KNO_3 strongly increased the electrophoretic velocity of the sol. A maximum set in at about 3 m. mol/liter. At higher concentrations the velocity again became smaller (Fig. 9).

This has been frequently observed. We have to deal here with a relaxation effect. Owing to the fact that there are so few ions in the solution, the layer of counter ions of the particles cannot be replenished sufficiently rapidly during electrophoresis, so that the particles are inhibited electrostatically by the counter ions moving in the opposite direction. Only little electrolyte is required to neutralize this effect.

The same effect was observed on a positive hydrophobic Fe_2O_3 sol. Here the electrophoretic velocity was about doubled by 5 m. mol KNO_3.

2°. $BaNO_3$. The Ba''-ion had such a preponderantly lowering influence that here no maximum was found.

3°. K_2SO_4 and $CH(SO_3K)_3$. In these salts again a maximum set in. At higher concentrations the electrophoretic velocity was not so strongly lowered by K_2SO_4 as by KNO_3. The influence of $CH(SO_3K)_3$ was still smaller.

Conc.	KNO_3	K_2SO_4	$CH(SO_3K)_3$
0 m. equiv./l.	—1.22 μ/sec/V/cm	—1.22 μ/sec/V/cm	—1.22 μ/sec/V/cm
20 „ „ „	—1.27 „	(—1.17) „	—1.57 „
30 „ „ „	—0.88 „	—1.01 „	—1.57 „

The K^+-ions compress the unknown double layer equally strongly in all salts. The ions of higher valency SO_4'' and $CH(SO_3)_3'''$, however, compress the normal double layer much more strongly than the NO_3'-ion, so that altogether the electrophoretic velocity is not so strongly reduced. In this sol, consequently, a double rule of SCHULZE–HARDY is observed.

8. *Summary.*

A dilute AgBr sol becomes rapidly extinct. During a few hours a flocculation takes place, which slowly comes to an end, owing to increasing stability of the sol. Sols with excess of $AgNO_3$ adsorb Ag^+-ions, but finally become negative under the influence of a new double layer, the construction of which is yet unknown, on the crystal surfaces formed by internal recrystallization.

Mathematics. — *Une inégalité relative aux sommes de* WEYL. Par J. G. VAN DER CORPUT.

(Communicated at the meeting of May 20, 1939.)

Dans cet article X désigne un entier $\geqq 2$.
P désignant un entier et

$$f(y) = \frac{\alpha}{n!} y^n + \alpha_1 y^{n-1} + \ldots + \alpha_n$$

un polynôme à coefficients réels, chaque somme de la forme

$$\sum_{y=P+1}^{P+X} e^{2\pi i f(y)}$$

s'appelle une somme de WEYL. Dans la théorie analytique des nombres nous avons besoin d'une borne supérieure pour le module de

$$S = X^{-1} \sum_{y=P+1}^{P+X} e^{2\pi i f(y)}.$$

Supposons que n *soit* > 1, *que* τ *soit un nombre* $\geqq 1$ *et* $\frac{a}{q}$ *une fraction irréductible à dénominateur positif, tels que nous ayons* $\left| \alpha - \frac{a}{q} \right| \leqq \frac{\tau}{q^2}$. Comme il est connu, dans ces conditions à tout nombre positif ε correspond un nombre c_1, dépendant uniquement de ε et n tel que

$$|S|^{2^{n-1}} < c_1 X^{\varepsilon} \left(\tau + \frac{q \lg q}{X} \right) \left(\frac{1}{q} + \frac{1}{X^{n-1}} \right). \quad . \quad . \quad . \quad (1)$$

Cette formule, qui a été appliquée à plusieurs reprises dans la théorie additive des nombres, n'a pas d'intérêt quand l'ordre de grandeur de q est moins élevée que celui de chaque puissance de X à exposant fixe; en effet, dans ce cas le membre de droite de (1) tend vers l'infini avec X. C'est précisément ce désavantage qui rend jusqu'à présent la formule inutile dans la théorie additive des nombres premiers. Cette difficulté provient du facteur X^{ε}. M. VINOGRADOW [1]) a remplacé l'inégalité par une autre, qui ne contient pas ce facteur X^{ε}, mais sa formule ne contient pas (1). Dans cet article je déduirai une inégalité qui est plus précise que (1) et en même temps plus précise que la formule de VINOGRADOW, à savoir:

---

[1]) Einige allgemeine Primzahlsätze. Travaux de l'Institut mathématique de Tbilissi, **3**, 35—67 (1938); voir p. 36, Hilfssatz 2.

Dans les conditions citées, à tout nombre positif $\varepsilon \leqq \frac{1}{2}$ *correspond un nombre* c_2, *dépendant uniquement de* ε *et de* n, *tel qu'on ait*

$$|S|^{2^{n-1}} < c_2\, x^{\sigma} \left(\tau + \frac{q}{X}\right)^{1-\varepsilon} \left(\frac{1}{q} + \frac{1}{X^{n-1}}\right)^{1-\varepsilon}; \quad . \; . \; . \quad (2)$$

dans cette formule on a

$$x = \log X \quad \text{et} \quad \sigma = \frac{(n-1)^l - 1}{l}, \; . \; . \; . \; . \; . \; . \quad (3)$$

où l *désigne le plus petit entier* $\geqq \dfrac{1}{\varepsilon}$.

Il est clair que (1) suit immédiatement de (2), même sans le facteur $\log q$; en effet, si $\sigma_1 = \dfrac{(n-1)^{l_1} - 1}{l_1}$, où l_1 est le plus petit entier $\geqq \dfrac{1}{2\varepsilon}$, on a $x^{\sigma_1} < c_3 X^{\frac{1}{2}\varepsilon}$, où c_3 ne dépend que de ε et de n (de même c_4 plus loin) et

$$\left(\tau + \frac{q}{X}\right)^{-\frac{1}{2}\varepsilon} \left(\frac{1}{q} + \frac{1}{X^{n-1}}\right)^{-\frac{1}{2}\varepsilon} < X^{\frac{1}{2}\varepsilon},$$

de sorte que la formule (2), appliquée avec $\frac{1}{2}\varepsilon$ au lieu de ε, donne

$$|S|^{2^{n-1}} < c_4\, X^{\varepsilon} \left(\tau + \frac{q}{X}\right) \left(\frac{1}{q} + \frac{1}{X^{n-1}}\right).$$

Dans la théorie additive des nombres premiers on prend jusqu' à présent pour q une valeur telle qu'à tout nombre positif fixe Ω correspond une constante c_6, dépendant de Ω, avec $q \geqq c_6\, x^{\Omega}$. Dans ce cas (1) n'a pas d'intérêt, mais (2) nous donne un nombre c_7, dépendant uniquement de $\varepsilon \leqq \frac{1}{2}$, n et Ω, tel qu'on ait, si τ est borné,

$$|S|^{2^{n-1}} < c_7\, x^{\sigma - (1-\varepsilon)\Omega}.$$

Désignons par Ω_1 un nombre fixe quelconque. Si nous prenons $\varepsilon = \frac{1}{2}$ (donc $l = 2$ et $\sigma = \frac{1}{2} n^2 - n$) et $\Omega = 2\sigma + 2^n \Omega_1$, nous trouvons un nombre c_8, dépendant uniquement de n et de Ω_1, tel que nous ayons

$$|S| < c_8\, x^{-\Omega_1}.$$

Cette inégalité, qui résulte d'ailleurs aussi de la formule de VINOGRADOW, nous apprend que S tend vers zéro, si X croît indéfiniment.

Pour la démonstration de (2) j'ai besoin de quelques lemmes.

Lemme 1: *Si* $u_1, \ldots, u_r, v_1, \ldots, v_s$ ($r \geqq 1, s \geqq 1$) *désignent des nombres naturels avec* $u_1 \ldots u_r = v_1 \ldots v_s$, *il existe* rs *nombres naturels* $a_{\varrho\sigma}$ *avec*

$$u_\varrho = a_{\varrho 1} \ldots a_{\varrho s}; \quad v_\sigma = a_{1\sigma} \ldots a_{r\sigma} \qquad (\varrho = 1, \ldots, r; \sigma = 1, \ldots, s).$$

Démonstration: Le lemme est évident dans le cas particulier $r = 1$. Supposons que r soit $\geqq 2$ et que le lemme ait été déjà démontré, si r est remplacé par $r - 1$.

Si nous désignons par (u, v) le plus grand commun diviseur de u et de v, et si nous posons

$$(u_1, v_1) = a_{11}; \quad \left(\frac{u_1}{a_{11}}, v_2\right) = a_{12}; \quad \left(\frac{u_1}{a_{11}\, a_{12}}, v_3\right) = a_{13};$$

$$\ldots; \left(\frac{u_1}{a_{11} \ldots a_{1,s-1}}, v_s\right) = a_{1s},$$

$$u_1 = a_{11} \ldots a_{1s}\, w \quad \text{et} \quad v_\sigma = a_{1\sigma}\, b_\sigma \qquad (\sigma = 1, \ldots, s),$$

nous avons $w u_2 \ldots u_r = b_1 \ldots b_s$. Le nombre naturel w étant premier avec $b_1 \ldots b_s$ est donc 1 et le lemme à démontrer, appliqué avec $r - 1$ au lieu de r, nous donne $(r-1)s$ entiers $a_{\varrho\sigma}$ avec

$$u_\varrho = a_{\varrho 1} \ldots a_{\varrho s} \quad \text{et} \quad b_\sigma = a_{2\sigma} \ldots a_{r\sigma} \qquad (\varrho = 2, \ldots, r; \sigma = 1, \ldots, s).$$

Le lemme est ainsi démontré.

Lemme 2: *Supposons que* l *désigne un entier* > 1, *et que* m *soit un nombre naturel. Si* $u_{\lambda\mu}$ ($\lambda = 1, \ldots, l$; $\mu = 1, \ldots, m$) *sont des nombres naturels tels que les* l *produits* $u_{\lambda 1} u_{\lambda 2} \ldots u_{\lambda m}$ ($\lambda = 1, \ldots, l$) *soient égaux l'un à l'autre, il existe* m^l *nombres naturels* $a_{\mu_1 \mu_2 \ldots \mu_l}$ *avec*

$$u_{\lambda\mu} = \Pi\, a_{\mu_1, \ldots, \mu_{\lambda-1}, \mu, \mu_{\lambda+1}, \ldots, \mu_l};$$

le produit est étendu aux nombres naturels $\mu_1, \ldots, \mu_{\lambda-1}, \mu_{\lambda+1}, \ldots, \mu_l$ *tous* $\leqq m$.

Démonstration: Le lemme précédent nous fournit le cas particulier $l = 2$. Supposons que l soit $\geqq 3$ et que le lemme ait été déjà démontré si l est remplacé par $l - 1$. De cette manière nous trouvons m^{l-1} nombres naturels $b_{\mu_1 \mu_2 \ldots \mu_{l-1}}$ avec

$$u_{\lambda\mu} = \Pi\, b_{\mu_1, \ldots, \mu_{\lambda-1}, \mu, \mu_{\lambda+1}, \ldots, \mu_{l-1}}$$

($\lambda = 1, \ldots, l-1$; $\mu = 1, \ldots, m$); le produit est étendu aux nombres naturels $\mu_1, \ldots, \mu_{\lambda-1}, \mu_{\lambda+1}, \ldots, \mu_{l-1}$, tous $\leqq m$. Nous avons donc

$$u_{l1}\, u_{l2} \ldots u_{lm} = \Pi\, b_{\mu_1 \ldots \mu_{l-1}},$$

où le produit figurant dans le membre de droite est étendu aux nombres naturels $\mu_1, \ldots, \mu_{l-1}$, tous $\leqq m$. D'après le lemme précédent, appliqué avec $r = m$ et $s = m^{l-1}$, il existe m^l nombres $a_{\mu_1 \ldots \mu_l}$ avec

$$b_{\mu_1 \ldots \mu_{l-1}} = \prod_{\mu_l = 1}^{m} a_{\mu_1 \ldots \mu_l} \quad \text{et} \quad u_{l\,\mu} = \Pi\, a_{\mu_1 \ldots \mu_{l-1}\,\mu};$$

le dernier produit est étendu aux nombres naturels $\mu_1, \ldots, \mu^l{}_{-1}$, tous $\leqq m$. Le lemme est donc démontré.

Lemme 3: *A tout nombre naturel L correspond un nombre c_9, tel que pour tout entier $X \geqq 2$ le nombre des systèmes formés par L nombres naturels $b_1, \ldots, b_L$ tels que*

$$b_1\, b_2 \ldots b_L \leqq X, \qquad \ldots\ldots\ldots\ldots \qquad (4)$$

est inférieur à $c_9\, X\, x^{L-1}$.

Démonstration: Le cas particulier $L = 1$ est évident. Supposons que L soit $\geqq 2$ et que le lemme ait été déjà démontré, si l'on remplace L par $L - 1$. L'entier b_1 parcourt le système des nombres naturels $\leqq X$. Si b_1 est choisi, le nombre des systèmes $b_2, \ldots, b_L$ qui entrent en considération est d'après le lemme à démontrer (appliqué avec $L - 1$ au lieu de L) inférieur à $c_{10}\, X\, b_1^{-1}\, x^{L-2}$, où c_{10} ne dépend que de L. Par conséquent le nombre des systèmes formés par L entiers $b_1, \ldots, b_L$ avec (4) est inférieur à

$$c_{10}\, X\, x^{L-2} \sum_{b_1 \leq X} \frac{1}{b} < c_9\, X\, x^{L-1}.$$

Lemme 4: *Si $\tau_m(h)$ désigne pour chaque choix des nombres naturels m et h le nombre de manières différentes d'écrire h comme le produit de m nombres naturels, à tout entier $l \geqq 2$ correspond un nombre c_{11}, dépendant uniquement de m et de l, tel qu'on ait pour tout entier $X \geqq 2$*

$$\sum_{h=1}^{X} \tau_m^l(h) < c_{11}\, X\, x^{m^l - 1}. \qquad \ldots\ldots\ldots\ldots \qquad (5)$$

Démonstration: $\tau_m(h)$ est le nombre des systèmes formés par $l\,m$ nombres naturels $u_{\lambda\mu}$ tels qu'on ait

$$u_{\lambda 1}\, u_{\lambda 2} \ldots u_{\lambda m} = h \qquad (\lambda = 1, \ldots, l).$$

Le membre de gauche de (5) est donc égal au nombre des systèmes, formés par $l\,m$ nombres naturels $u_{\lambda\mu}$ tels que $\prod_{\mu=1}^{m} u_{\lambda\mu}$ soit indépendant de λ et en outre $\leqq X$. A chacun de ces systèmes correspond, d'après le lemme 2, un système formé par m^l entiers $a_{\mu_1 \ldots \mu_l}$ vérifiant l'inégalité

$$\prod_{\mu_1=1}^{m} \ldots \prod_{\mu_l=1}^{m} a_{\mu_1 \ldots \mu_l} \leqq X; \qquad \ldots\ldots\ldots\ldots \qquad (6)$$

à deux systèmes différents $(u_{\lambda\mu})$ correspondent deux systèmes $(a_{\mu_1\ldots\mu_l})$, qui sont différents l'un de l'autre. Par conséquent le membre de gauche de (5) est au plus égal au nombre des systèmes formés par m^l nombres naturels $a_{\mu_1\ldots\mu_l}$ avec la propriété (6), donc, d'après le lemme précédent, inférieur à $c_{11}\, X\, x^{m^l-1}$.

Lemme 5: *Supposons que le nombre réel* α, *le nombre* $\tau \geqq 1$ *et la fraction irréductible* $\frac{a}{q}$ *soient tels qu'on ait* $\left|\alpha - \frac{a}{q}\right| \leqq \frac{\tau}{q^2}$ *et posons pour tout nombre naturel* h

$$M(h) = min\left(1, \frac{1}{X\{h\alpha\}}\right),$$

où $\{u\}$ *désigne la distance de* u *à l'entier le plus voisin; min* (u, v) *désigne le plus petit des deux nombres* u *et* v.

A tout nombre $\varrho > 1$ *correspond un nombre* c_{12}, *dépendant uniquement de* ϱ, *tel qu'on ait pour tout entier* D

$$\sum_{h=D+1}^{D+q} M^{\varrho}(h) < c_{12}(\tau + X^{-1}q). \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Démonstration: Si l'on pose $C = D + \frac{1}{2}q$ ou $D + \frac{1}{2}(q+1)$, selon que q est pair ou impair, on a $|h - C| \leqq \frac{1}{2}q$ pour chaque $h \geqq D+1$ et $\leqq D+q$, par conséquent en vertu de $\left|\alpha - \frac{a}{q}\right| \leqq \frac{\tau}{q^2}$,

$$\left|\alpha h - \alpha C - \frac{a}{q}(h - C)\right| \leqq \frac{\tau\,|h - C|}{q^2} \leqq \frac{\tau}{2q}. \quad . \quad . \quad . \quad (8)$$

Si h parcourt le système $D+1, \ldots, D+q$, le produit $a(h-C)$ parcourt un système complet de restes modulo q, parce que a est premier avec q. Désignons par r_h le reste modulo 1 de $\alpha C + \frac{a}{q}(h - c)$, tel que nous ayons $-\frac{1}{2} < r_h \leqq \frac{1}{2}$. Ces points $r_h\,(h = D+1, \ldots, D+q)$ sont équidistants et la distance vaut $\frac{1}{q}$. Le nombre des h avec $|r_h| \leqq \frac{1}{q}(\frac{1}{2}\tau + 1)$ est donc tout au plus $1 + 2(\frac{1}{2}\tau + 1) \leqq 4\tau$ et la contribution de ces h au membre de gauche de (7) est $\leqq 4\tau$.

A chaque nombre naturel g correspond tout au plus un h avec

$$\frac{1}{q}(\tfrac{1}{2}\tau + g) < r_h \leqq \frac{1}{q}(\tfrac{1}{2}\tau + g + 1)$$

et pour cet h éventuel on a

$$\{\alpha h\} \geqq r_h - q \cdot \frac{\tau}{2q^2} > \frac{g}{q}.$$

La contribution des h avec $r_h > \frac{1}{q}(\frac{1}{2}\tau + 1)$ est donc inférieure à

$$\sum_{g=1}^{\infty} \min(1, X^{-\varrho} q^{\varrho} g^{-\varrho}) \leqq \sum_{g \leqq \tau + X^{-1} q} 1 + \sum_{g > \tau + X^{-1} q} X^{-\varrho} q^{\varrho} g^{-\varrho}$$
$$< c_{13} (\tau + X^{-1} q),$$

où c_{13} dépend uniquement de ϱ. On trouve la même borne supérieure pour la contribution des h avec $r_h < -\frac{1}{q}(\frac{1}{2}\tau + 1)$ et le lemme est donc démontré.

Lemme 6: *Dans les conditions du lemme précédent on a pour tout nombre naturel H et pour tout nombre $\varrho > 1$*

$$H^{-1} \sum_{h=1}^{H} M^{\varrho}(h) < c_{14} (\tau + X^{-1} q)(q^{-1} + H^{-1}),$$

où c_{14} ne dépend que de ϱ.

Démonstration: Partageons la somme $\sum_{h=1}^{H}$ en sommes partielles contenant chacune au plus q termes, de telle façon que le nombre de ces sommes partielles soit inférieur à $\frac{H}{q} + 1$. Le lemme précédent, appliqué à chacune de ces sommes partielles, donne le lemme 6.

Démonstration de (2).

Comme on le sait [1]), on a

$$|S|^{2^{n-1}} < 4^{2^{n-1}} \left(X^{-1} + X^{1-n} \sum_{h_1, \ldots, h_{n-1}=1}^{X} \min\left(1, \frac{1}{X\{\alpha h_1 \ldots h_{n-1}\}}\right)\right)$$
$$\leqq 4^{2^{n-1}} \left(X^{-1} + X^{1-n} \sum_{h=1}^{X^{n-1}} \tau_{n-1}(h) \min\left(1, \frac{1}{X\{\alpha h\}}\right)\right),$$

où $\tau_{n-1}(h)$ désigne le nombre de manières d'écrire h comme le produit de $n-1$ nombres naturels. L'inégalité de CAUCHY généralisée nous donne donc pour tout entier $l \geqq 2$

$$|S|^{2^{n-1}} < 4^{2^{n-1}} \left(X^{-1} + A^{\frac{1}{l}} B^{\frac{l-1}{l}}\right),$$

où

$$A = X^{1-n} \sum_{h=1}^{X^{n-1}} \tau_{n-1}^{l}(h) < c_{15}\, x^{(n-1)^{l}-1}$$

---

[1]) Comparer par exemple E. LANDAU, Vorlesungen über Zahlentheorie, I, p. 253 (Satz 265).

d'après le lemme 4 et

$$B = X^{1-n} \sum_{h=1}^{X^{n-1}} min\left(1, \frac{1}{X^{\frac{l}{l-1}} \{\alpha h\}^{\frac{l}{l-1}}}\right)$$

$$< c_{16} (\tau + X^{-1} q) (q^{-1} + X^{1-n})$$

d'après le lemme précédent (c_{15}, c_{16} et c_{17} ne dépendent que de n et de l). En vertu de

$$X^{-1} < (\tau + X^{-1} q) (q^{-1} + X^{1-n})$$

on trouve donc

$$|S|^{2^{n-1}} < c_{17}\, x^{\frac{(n-1)^{l}-1}{l}} (\tau + X^{-1} q)^{1-\frac{1}{l}} (q^{-1} + X^{1-n})^{1-\frac{1}{l}},$$

d'où suit (2), si nous prenons pour l le plus petit entier $\geqq \frac{1}{\varepsilon}$.

Mathematics. — *Sur la méthode des points décisifs.* Par J. G. VAN DER CORPUT.

(Communicated at the meeting of May 20, 1939.)

Il y a quelques années j'ai publié la méthode des points décisifs dans la Compositio Mathematica[1]). Dans le présent article, dans lequel je ne traiterai qu'un cas très particulier, je ferai quelques remarques qui seront compréhensibles aussi pour le lecteur qui ne connaît pas encore la méthode. Elles lui permettront de trouver pour beaucoup d'intégrales non seulement une valeur approximative, mais aussi une borne supérieure numérique pour l'écart. La méthode peut être appliquée aux intégrales de la forme

$$\int_a^b g(u)\cos\varphi(u)\,.\,du \quad \text{et} \quad \int_a^b g(u)\sin\varphi(u)\,.\,du \quad .\ .\ .\ .\ \quad (1)$$

et même aux intégrales plus générales

$$I = \int_C g(w)\, e^{f(w)}\, dw,$$

où C désigne une courbe continue rectifiable située dans le plan complexe des w (une des extrémités où toutes les deux peuvent être à l'infini). Considérons la dernière intégrale. Supposons que $g(w)$ soit dérivable le long de C, c'est-à-dire que pour tout point w de C la fraction $\dfrac{g(z)-g(w)}{z-w}$ tende vers une limite finie, si le point $z \neq w$ situé sur C tend vers w. Je supposerai que cette limite, que je désignerai par $g'(w)$, soit continue le long de C. Supposons en outre que la dérivée troisième de $f(w)$, prise le long de C, existe et soit continue et qu'enfin la condition suivante, dans laquelle $\sqrt{f''(w)}$ désigne une fonction continue de w le long de C, soit vérifiée:

La courbe C ne contient aucun point w tel que $\dfrac{f'(w)}{\sqrt{f''(w)}}$ soit un nombre réel $\neq 0$ et elle contient au plus un nombre fini de points ζ avec $f'(\zeta) = 0$. La dérivée $f''(\zeta)$ n'est $\geqq 0$ en aucun point ζ de C avec $f'(\zeta) = 0$.

Un point ζ de C, ne coïncidant avec aucune des extrémités de C, est appelé un point décisif, lorsque $f'(\zeta)=0$ et que $\Im \dfrac{f'(w)}{\sqrt{f(w)}}$ a des signes opposés 2) de part et d'autre de ζ sur C.

---

[1]) Zur Methode der stationären Phase II. Compositio Mathematica **3**, 328—372 (1936).

Si l'on pose $f(u)=i\,\varphi(u)$ et si l'on prend pour C le segment linéaire (a, b), on obtient les intégrales figurant dans (1) et la méthode des points décisifs revient à celle de la phase stationnaire; dans ce cas la phase $\varphi(u)$ est stationnaire aux points décisifs.

La méthode des points décisifs repose essentiellement sur l'intégration par parties. Afin de pouvoir appliquer l'intégration par parties je définis pour chaque point w de C, qui ne coïncide pas avec un point décisif, certaines fonctions $\psi_0(w)=e^{f(w)}$, $\psi_1(w)$, $\psi_2(w), \ldots$ qui possèdent les propriétés suivantes:

I. Pour chaque point w de C, qui n'est pas un point décisif et pour chaque $\sigma \geqq 0$ la fonction $\psi_{\sigma+1}(w)$ est dérivable le long de C et on a 3)

$$\psi_\sigma(w) = -\psi'_{\sigma+1}(w) + \tfrac{1}{2}(\sigma+1)(\sigma+2)\, f'''(w)\, \psi_{\sigma+3}(w) \quad . \quad . \quad (2)$$

et en outre pour tout $\sigma \geqq 1$

$$|\psi_\sigma(w)| < \frac{\sigma\, 2^{\sigma+\frac{1}{2}}}{|f'(w)|^\sigma}\, |e^{f(w)}| \text{ et } < \frac{2^{\frac{1}{2}\sigma+\frac{1}{2}}\, \Gamma\left(\dfrac{\sigma}{2}\right)}{(\sigma-1)!\, |f''(w)|^{\frac{1}{2}\sigma}}\, |e^{f(w)}|. \quad . \quad (3)$$

II. Si un point w situé sur C tend vers un point ζ de C avec $f'(\zeta)=0$ de telle façon que $\Im \dfrac{f'(w)}{\sqrt{f''(w)}}$ possède constamment un signe fixe, $\psi_\sigma(w)$ tend pour chaque nombre naturel σ vers

$$\frac{2^{\frac{1}{2}\sigma-1}\, e^{\sigma i\left(\pm\frac{\pi}{2}-\gamma\right)}\, \Gamma\left(\frac{1}{2}\sigma\right) e^{f(\zeta)}}{(\sigma-1)!\, |f''(\zeta)|^{\frac{1}{2}\sigma}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

on doit prendre le signe ±, selon que $\Im \dfrac{f'(w)}{\sqrt{f''(w)}}$ est constamment positif ou constamment négatif; on a $\gamma = \arg \sqrt{f''(\zeta)}$.

Ces remarques nous permettront de mettre l'intégrale

$$I_\sigma = \int_C g(w)\, \psi_\sigma(w)\, dw,$$

qui prend pour $\sigma=0$ la valeur I, sous une autre forme. La formule (2) nous apprend

$$I_\sigma = -\int_C g(w)\, \psi'_{\sigma+1}(w)\, dw + \tfrac{1}{2}(\sigma+1)(\sigma+2) \int_C f'''(w)\, g(w)\, \psi_{\sigma+3}(w)\, dw,$$

---

2) Si $z=x+iy$, où x et y sont réels, on écrit $x=\Re z$ et $y=\Im z$.

3) Le raisonnement figurant à la page 341 de l'article cité dans le renvoi 1) nous fournit les inégalités (3) avec les facteurs 8^σ et $24^{\frac{1}{2}\sigma}$, au lieu de $2^{\sigma+\frac{1}{2}}$ et $2^{\frac{1}{2}\sigma+\frac{1}{2}}$.

d'où nous obtenons en integrant par parties

$$\left.\begin{aligned} I_\sigma = -\Big|_C\, g(w)\,\psi_{\sigma+1}(w) + \int_C g'(w)\,\psi_{\sigma+1}(w)\,dw \\ + \tfrac{1}{2}(\sigma+1)(\sigma+2)\int_C f'''(w)\,g(w)\,\psi_{\sigma+3}(w)\,dw. \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

Le premier terme du membre de droite est la somme des contributions des extrémités de C et de celles des points décisifs. La relation (4) nous fournit les contributions des points décisifs. Cette formule nous donne aussi celle d'une extrémité, dans le cas particulier où la dérivée de f s'y annule. Si ζ est une extrémité de C, située à l'infini, et si

$$\frac{g(w)\,e^{f(w)}}{|f'(w)|^{\sigma+1} + |f''(w)|^{\frac{1}{2}\sigma+\frac{1}{2}}} \to 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

w parcourant la courbe C et tendant vers ζ, la contribution de cette extrémité ζ vaut zéro en vertu de (3).

Au besoin on peut répéter l'intégration par parties, si du moins les fonctions $f(w)$ et $g(w)$ sont dérivables le long de C un nombre suffisant de fois. On peut écrire ainsi I_σ comme la somme de quelques termes calculables d'une manière élémentaire et de plusieurs intégrales de la forme

$$\int_C \chi(w)\,\psi_\tau(w)\,dw;$$

(3) donne pour la valeur absolue de ces dernières intégrales une borne supérieure numérique.

Passons maintenant à la démonstration. A tout point w de C j'adjoins l'ensemble des points z avec

$$\Im\{f'(w)(z-w) + \tfrac{1}{2} f''(w)(z-w)^2\} = 0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Cet ensemble est une droite ou une hyperbole équilatère, selon que $f''(w)$ est égal à zéro ou non. Si $f''(w)$ vaut zéro, je désigne par $W(w)$ la partie de cette droite formée par les points z avec

$$f'(w)(z-w) + \tfrac{1}{2} f''(w)(z-w)^2 \leqq 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Lorsque $f''(w) \neq 0$ et que w n'est pas un point décisif, je distingue deux cas différents:

1. Traitons d'abord le cas où $\dfrac{(f'(w))^2}{f''(w)}$ n'est pas réel. L'hyperbole équilatère définie par (7) ne contient pas son centre z_0 qui est défini par

$$f'(w) + f''(w)(z_0 - w) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

L'hyperbole équilatère n'est donc pas dégénérée et le point w est situé sur une seule branche de l'hyperbole. Par $W(w)$ je désigne la partie de cette branche qui est formée par les points z avec (8).

2. Traitons ensuite le cas où la fraction $\frac{(f'(w))^2}{f''(w)}$ est réelle. Les conditions imposées à la fonction $f(w)$ nous apprennent que cette fraction n'est pas >0, de sorte qu'elle est négative ou nulle. Si elle est négative, l'hyperbole équilatère définie par (7) est formée par deux droites d et d', dont une seule contient w. Soit d la droite contenant w. Je désigne par $W(w)$ la partie de d qui est limitée au point w et qui ne contient pas le point d'intersection z_0 des deux droites d et d'. Ce point z_0 vérifie (9) et donc aussi

$$f'(w)(z-w)+\tfrac{1}{2}f''(w)(z-w)^2=-\frac{(f'(w))^2}{2f''(w)}>0,$$

de sorte que (8) vaut par chaque point z de $W(w)$. Si $f'(w)=0$ et si w n'est pas un point décisif, un point w_1 de C tendant vers w possède la propriété que $W(w_1)$ tend vers une demi-droite (bornée par w) que j'appelle $W(w)$.

La courbe $W(w)$ étant ainsi définie d'une manière univoque pour tout point w de C ne coïncidant pas avec un point décisif, j'introduis les fonctions $\psi_\sigma(w)$ $(\sigma \geqq 1)$ par la définition

$$\psi_\sigma(w)=\frac{1}{(\sigma-1)!}\int_{w}^{\infty}(z-w)^{\sigma-1}\,e^{f(w)+f'(w)(z-w)+\frac{1}{2}f''(w)(z-w)^2}\,dz,$$

où l'intégrale est prise le long de $W(w)$.

Démonstration de (2).

Posons pour tout entier $\sigma \geqq 0$

$$F_\sigma(w,z)=\frac{1}{\sigma!}(z-w)^\sigma\,e^{p(w,z)},$$

où

$$p(w,z)=f(w)+f'(w)(z-w)+\tfrac{1}{2}f''(w)(z-w)^2.$$

Si w désigne un point de C ne coïncidant pas avec un point décisif et si $h\neq 0$ est en valeur absolue assez petit, $w+h$ désignant un point de C, on a

$$\int_{w+h}^{\infty}F_\sigma(w+h,z)\,dz-\int_{w}^{\infty}F_\sigma(w+h,z)\,dz=-\int_{w}^{w+h}F_\sigma(w+h,z)\,dz.$$

En vertu de

$$F_0(w,w)=e^{f(w)}=\psi_0(w)\ \text{ et }\ F_\sigma(w,w)=0 \qquad (\sigma\geqq 1)$$

on obtient pour $h\to 0$

$$\frac{1}{h}\int_{w+h}^{\infty}F_\sigma(w+h,z)\,dz-\frac{1}{h}\int_{w}^{\infty}F_\sigma(w+h,z)\,dz\to-\psi_0(w)\ \text{ ou }\ 0,$$

selon que σ est nul ou positif. On a

$$\frac{\partial F_\sigma(w, z)}{\partial w} = \left(\frac{1}{\sigma!} (z-w)^{\sigma+2} \tfrac{1}{2} f'''(w) - \frac{1}{(\sigma-1)!} (z-w)^{\sigma-1} \right) e^{p(w,z)};$$

pour $\sigma = 0$ le terme $\frac{1}{(\sigma-1)!} (z-w)^{\sigma-1}$ n'apparaît pas. Par conséquent on trouve pour $h \to 0$

$$\int_w^\infty \frac{F_0(w+h, z) - F_0(w)}{h} dz \to \tfrac{1}{2} f'''(w) \int_w^\infty (z-w)^2 e^{p(z,w)} dz = f'''(w) \psi_3(w)$$

et en outre pour chaque nombre naturel σ

$$\int_w^\infty \frac{F_\sigma(w+h, z) - F_\sigma(w)}{h} dz \to \tfrac{1}{2} (\sigma+1)(\sigma+2) f'''(w) \psi_{\sigma+3}(w) - \psi_\sigma(w).$$

Finalement on trouve donc

$$\frac{\psi_{\sigma+1}(w+h) - \psi_{\sigma+1}(w)}{h} = \frac{1}{h} \int_{w+h}^\infty F_\sigma(w+h, z)\, dz - \frac{1}{h} \int_w^\infty F_\sigma(w+h, z)\, dz$$

$$+ \int_w^\infty \frac{F_\sigma(w+h) - F_\sigma(w)}{h} dz$$

$$\to -\psi_\sigma(w) + \tfrac{1}{2} (\sigma+1)(\sigma+2) f'''(w) \psi_{\sigma+3}(w).$$

Démonstration de (3).

Pour la démonstration de (3), j'ai besoin d'un lemme.

Lemme: *Deux points $P_1(x_1, y_1)$ et $P_2(x_2, y_2)$ situés sur une même branche de l'hyperbole équilatère $x^2 - y^2 = c^2$ $(c > 0)$, vérifient les inégalités*

$$|x_2 y_2 - x_1 y_1| \geqq \tfrac{1}{2} OP_1 . P_1 P_2 . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

et

$$|x_2 y_2 - x_1 y_1| \geqq \tfrac{1}{2} P_1 P_2^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

où O désigne l'origine des coordonnées.

Remarque: Dans aucun des deux membres de droite le facteur $\frac{1}{2}$ ne peut être remplacé par une constante plus grande; en effet, si $x_2 = c$ et $y_2 = o$, les deux fractions

$$\frac{|x_2 y_2 - x_1 y_1|}{OP_1 . P_1 P_2} \quad \text{et} \quad \frac{|x_2 y_2 - x_1 y_1|}{P_1 P_2^2}$$

tendent vers $\frac{1}{2}$, lorsque x_1 croît indéfiniment.

Démonstration: Sans nuire à la généralité nous pouvons supposer que P_1 et P_2 sont situés à droite de l'axe des y. Si l'on pose

$$x = \tfrac{1}{2} c \, (e^{\alpha} + e^{-\alpha}) \text{ et } y = \tfrac{1}{2} c \, (e^{\alpha} - e^{-\alpha})$$

et si $\alpha = \beta$ et $\alpha = \gamma$ correspondent aux points P_1 et P_2, on a

$$\begin{aligned} x_2 \, y_2 - x_1 \, y_1 &= \tfrac{1}{4} c^2 \, (e^{2\gamma} - e^{-2\gamma} - e^{2\beta} + e^{-2\beta}) \\ &= \tfrac{1}{4} c^2 \, (e^{-2\beta} - e^{-2\gamma}) \, (e^{2\beta + 2\gamma} + 1), \\ O P_1^2 &= \tfrac{1}{2} c^2 \, (e^{2\beta} + e^{-2\beta}) \end{aligned}$$

et

$$\begin{aligned} P_1 \, P_2^2 &= \tfrac{1}{4} c^2 \, (e^{\gamma} + e^{-\gamma} - e^{\beta} - e^{-\beta})^2 + \tfrac{1}{4} c^2 \, (e^{\gamma} - e^{-\gamma} - e^{\beta} + e^{-\beta})^2 \\ &= \tfrac{1}{2} c^2 \, (e^{\gamma} - e^{\beta})^2 + \tfrac{1}{2} (e^{-\gamma} - e^{-\beta})^2 \\ &= \tfrac{1}{2} c^2 \, (e^{-\gamma} - e^{-\beta})^2 \, (e^{2\beta + 2\gamma} + 1). \end{aligned}$$

Par conséquent (10) découle de l'inégalité évidente

$$(e^{-\gamma} + e^{-\beta})^2 \, (e^{2\beta + 2\gamma} + 1) \geqq e^{2\beta} + e^{-2\beta}$$

et (11) résulte de

$$e^{-\gamma} + e^{-\beta} \geqq | \, e^{-\gamma} - e^{-\beta} \, |.$$

Afin d'obtenir (3) je démontrerai d'abord pour tout point w de C ne coïncidant pas avec un point décisif et pour tout point z de $W(w)$ les inégalités

$$\left.\begin{aligned} \Re \, \{ f'(w) \, (z - w) + \tfrac{1}{2} f''(w) \, (z - w)^2 \} &\leqq - \tfrac{1}{2} \, | f'(w) | \; | z - w | \\ \text{et } &\leqq - \tfrac{1}{2} \, | f''(w) | \; | z - w |^2. \end{aligned}\right\} . \qquad (12)$$

Ces inégalités sont évidentes, si $f''(w) = 0$, comme le membre de gauche est alors égal à $- | f'(w) | \; | z - w |$. Traitons ensuite le cas où $f''(w) \neq 0$ et que l'hyperbole équilatère définie par (7) est formée par deux droites. Sans nuire à la généralité on peut supposer que cette hyperbole soit formée par les axes réel et imaginaire et que w soit positif, sinon il suffit d'effectuer une translation et une rotation. Dans ce cas on a $f'(w) \leqq 0$ et $f''(w) \leqq 0$ et $W(w)$ est formé par les points $z \geqq w$, de sorte que les inégalités en question sont évidentes.

Traitons enfin le cas où $f''(w) \neq 0$ et où l'hyperbole equilatère définie par (7) n'est pas dégénérée. Sans nuire à la généralité on peut supposer que l'origine des coordonnées coïncide avec le centre de l'hyperbole et que l'axe réel coïncide avec l'axe focal de l'hyperbole; au besoin on effectue une translation et une rotation. Dans ce cas on a

$$\Im \, (f'(w) \, (z - w) + \tfrac{1}{2} f''(w) \, (z - w)^2) = \lambda \, (x^2 - y^2 - c^2),$$

où λ est réel et c positif, d'où il suit

$$f'(w) \, (z - w) + \tfrac{1}{2} f''(w) \, (z - w)^2 = i \, \lambda \, z^2 + \mu - i \, \lambda \, c^2$$

(μ réel), par conséquent

$$\Re\{f'(w)(z-w)+\tfrac{1}{2}f''(w)(z-w)^2\}=-2\lambda xy+\mu.$$

Si les coordonnées de w sont désignées par x_1 et y_1, on a donc $\mu=2\lambda x_1 y_1$ et

$$\begin{aligned}|\Re\{f'(w)(z-w)+\tfrac{1}{2}f''(w)(z-w)^2\}|&=2|\lambda|\;|xy-x_1y_1|\\ &\geqq|\lambda|\;|w|\;|z-w|\ \text{ et }\geqq|\lambda|\;|z-w|^2\end{aligned}$$

d'après le lemme précédent. En vertu de $f'(w)=2i\lambda w$ et $f''(w)=2i\lambda$ on obtient de cette manière les inégalités (12), dont le membre de gauche est $\leqq 0$.

Si l'on pose $|z-w|=\varrho$, on a pour tout point $z\neq w$ de $W(w)$, que $\left|\frac{d\varrho}{dz}\right|=\cos\varphi$, où φ est l'angle positif $<\frac{\pi}{4}$ formé au point z par l'hyperbole (7) et la droite joignant z à w; on a donc $\left|\frac{d\varrho}{dz}\right|<\frac{1}{2}\sqrt{2}$ et $|\psi_\sigma(w)|$ est pour chaque nombre naturel σ inférieur à

$$\frac{1}{(\sigma-1)!}\int_0^\infty \varrho^{\sigma-1}\,|e^{f(w)}|\,e^{-\frac{1}{2}|f'(w)|\varrho}\sqrt{2}\,d\varrho$$

et aussi inférieur à

$$\frac{1}{(\sigma-1)!}\int_0^\infty \varrho^{\sigma-1}\,|e^{f(w)}|\,e^{-\frac{1}{2}|f''(w)|\varrho^2}\sqrt{2}\,d\varrho,$$

d'où suit (3).

Démonstration de (4).

Sans nuire à la généralité nous pouvons supposer que le point w situé sur C tend vers ζ de telle façon que $\Im\frac{f'(w)}{\sqrt{f''(w)}}$ soit toujours positif; sinon il suffit de remplacer $\sqrt{f''(w)}$ par $-\sqrt{f''(w)}$ et γ par $\gamma+\pi$. Pour chaque point z de $W(w)$ on a

$$f'(w)(z-w)+\tfrac{1}{2}f''(w)(z-w)^2=-p, \quad . \quad . \quad . \quad . \quad (13)$$

où p est $\geqq 0$. Si w et p sont donnés, le point z sur $W(w)$, défini par (13), est déterminé univoquement et, si nous désignons cette valeur de z par $z_p(w)$, nous avons

$$z_p(w)=w-\frac{f'(w)}{f''(w)}+\frac{1}{\sqrt{f''(w)}}\chi_p(w),$$

où $\chi_p(w)$ est une fonction continue de la variable $p\geqq 0$, qui prend pour $p=0$ la valeur $\frac{f'(w)}{\sqrt{f''(w)}}$ et qui possède pour $p\geqq 0$ une des deux valeurs $\pm\sqrt{-2p+\frac{(f'(w))^2}{f''(w)}}$. Comme $\frac{f'(w)}{\sqrt{f''(w)}}$ n'est pas réel, le cas $\frac{(f'(w))^2}{f''(w)}\geqq 0$

est exclu et $-2p+\frac{(f'(w))^2}{f''(w)}$ n'est $\geqq 0$ pour aucune valeur $\geqq 0$ de p. L'argument de $\chi_p(w)$ n'est donc pas égal à un multiple de π. Comme $\chi_p(w)$ est une fonction continue de $p \geqq 0$ et possède pour $p=0$ la valeur $\frac{f'(w)}{\sqrt{f''(w)}}$, dont la partie imaginaire est positive, nous pouvons supposer que cet argument soit situé entre 0 et π. Si p est un nombre positif fixe, et si w tend sur C vers ζ, le nombre $\chi_p(w)$ tend vers le nombre purement imaginaire $\pm\sqrt{-2p}$, dont l'argument est donc égal à $+\frac{\pi}{2}$. Le point $z_p(w)$ tend donc vers un point

$$\zeta+\frac{i h_p}{\sqrt{f''(w)}},$$

où h_p est positif, de sorte que $W(w)$ se transforme en la droite qui joint les deux points ζ et $\zeta+\frac{i\infty}{\sqrt{f''(\zeta)}}$. Par conséquent $\psi_\sigma(w)$ tend pour chaque nombre naturel σ vers

$$\frac{1}{(\sigma-1)!}\int_{\zeta}^{\zeta+\frac{i\infty}{\sqrt{f''(\zeta)}}}(z-\zeta)^{\sigma-1}\,e^{f(\zeta)+\frac{1}{2}f''(\zeta)(z-\zeta)^2}\,dz$$

$$=\frac{2^{\frac{1}{2}\sigma-1}\,e^{\sigma i\left(\frac{\pi}{2}-\gamma\right)}\,\Gamma\left(\frac{1}{2}\sigma\right)e^{f(\zeta)}}{(\sigma-1)!\,|f''(\zeta)|^{\frac{1}{2}\sigma}},$$

où $\gamma=arg\sqrt{f''(w)}$.

Mathematics. — *Sur la discrépance modulo un.* (Première communication.)

Par J. G. VAN DER CORPUT et CH. PISOT.

(Communicated at the meeting of May 20, 1939.)

Soit U un système de n nombres réels $u_1, u_2, \ldots, u_n$. Nous attachons à chaque suite U un nombre $D(U)$ que nous appelons la *discrépance modulo un* de U et que nous définissons de la manière suivante:

Soient α et λ des nombres réels quelconques, nous désignons par $N(\alpha, \alpha+\lambda)$ le nombre des u_ν de la suite U compris modulo un entre α et $\alpha+\lambda$, c'est-à-dire de façon précise $N(\alpha, \alpha+\lambda)$ est le nombre des u_ν de la suite U qui vérifient les inégalités:

$$0 \leqq u_\nu - \alpha - [u_\nu - \alpha] < \lambda - [\lambda],$$

où $[a]$ désigne l'entier de a.

Posons

$$N(\alpha, \alpha+\lambda) = (\lambda - [\lambda] + D(\alpha, \alpha+\lambda))\, n.$$

Par définition $D(U)$ est le plus petit nombre tel que l'on ait:

$$|D(\alpha, \alpha+\lambda)| \leqq D(U)$$

pour toutes les valeurs de α et de λ.

Remarquons que $N(\alpha, \alpha+\lambda)$ et $D(\alpha, \alpha+\lambda)$ sont des fonctions périodiques de période 1 en α et en λ, nulles pour $\lambda = 0$. Comme $N(\alpha, \alpha+\lambda) \leqq n$, on a $|D(\alpha, \alpha+\lambda)| \leqq 1$ et par suite $D(U) \leqq 1$. D'autre part étant donné un nombre ε aussi petit que l'on veut, il y a toujours au moins un nombre α tel que $N(\alpha, \alpha+\varepsilon) \geqq 1$, donc pour lequel $D(\alpha, \alpha+\varepsilon) \geqq \frac{1-\varepsilon}{n}$; on a donc aussi $D(U) \geqq \frac{1}{n}$.

Enfin la fonction $D(\alpha, \beta)$ possède la propriété additive suivante: Quels que soient les nombres α, β, γ, on a

$$D(\alpha, \beta) + D(\beta, \gamma) = D(\alpha, \gamma).$$

En effet, en vertu de la périodicité en β de $D(\alpha, \beta)$ et en γ de $D(\beta, \gamma)$, nous pouvons supposer que

$$\alpha \leqq \beta < \alpha + 1 \quad \text{et} \quad \beta \leqq \gamma < \beta + 1,$$

de sorte que $\alpha \leqq \gamma < \alpha + 2$. Dans ces conditions

$$N(\alpha, \beta) = n(\beta - \alpha + D(\alpha, \beta)), \; N(\beta, \gamma) = n(\gamma - \beta + D(\beta, \gamma));$$
$$N(\alpha, \gamma) = n(\gamma - \alpha + D(\alpha, \gamma)) \text{ si } \alpha \leqq \gamma < \alpha + 1$$

et

$$N(\alpha, \gamma) = n\,(\gamma - \alpha - 1 + D(\alpha, \gamma)) \text{ si } \alpha + 1 \leqq \gamma < \alpha + 2.$$

Or si

$$\alpha \leqq \gamma < \alpha + 1, \text{ on a } N(\alpha, \beta) + N(\beta, \gamma) = N(\alpha, \gamma)$$

et si

$$\alpha + 1 \leqq \gamma < \alpha + 2, \text{ on a } N(\alpha, \beta) + N(\beta, \gamma) = n + N(\alpha, \gamma),$$

la propriété est donc démontrée.

En particulier pour $\gamma = \alpha$ on obtient la relation $D(\alpha, \beta) = -D(\beta, \alpha)$ valable pour tous les nombres α et β.

Désignons par U_2 le système des n^2 nombres $u_\nu - u_\varrho$ ($\nu = 1, 2, \ldots, n$; $\varrho = 1, 2, \ldots, n$). M. VINOGRADOW [1]) a montré que si les n nombres $nu_1, \ldots, nu_n$ sont entiers, la connaissance de $D(U_2)$ permet de trouver une borne supérieure pour $D(U)$, à savoir:

$$D(U) < C \sqrt[3]{D(U_2)}$$

où C est une constante absolue convenable.

VAN DER CORPUT a amélioré le résultat de M. VINOGRADOW en démontrant l'inégalité plus forte

$$D(U) < 2^{5 + \sqrt{2\,|\log D(U_2)}} \sqrt{D(U_2)}$$

et cela sans supposer les nombres $nu_1, \ldots, nu_n$ entiers [2]). Dans cette communication nous démontrerons dans les mêmes conditions l'inégalité plus précise

$$D(U) < 2^{\frac{7}{2} + \sqrt{\frac{|\log D(U_2)|}{\log 2}}} \sqrt{D(U_2)}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Nous en déduirons l'inégalité équivalente

$$D(U) < 2^{\frac{7}{2} + \frac{1}{4\varepsilon}} (D(U_2))^{\frac{1}{2} - \varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

valable pour tout $\varepsilon > 0$. L'inégalité de M. VINOGRADOW s'obtient en prenant $\varepsilon = \frac{1}{6}$ dans l'inégalité (2). On trouve ainsi que 2^5 est une valeur possible pour la constante absolue C.

Comme nous le démontrerons dans les communications suivantes, le même raisonnement permet d'obtenir pour la répartition modulo un des polynômes les mêmes résultats que par la méthode de WEYL et de baser la théorie de la répartition uniforme modulo un sur des considérations élémentaires.

---

[1]) Ueber die Bruchteile eines ganzen Polynoms. Bull. Acad. Sci. U.R.S.S., (6) T. 20, p. 585—600 (1926).

[2]) La démonstration de cette inégalité n'a pas encore paru, mais le résultat a déjà été publié par J. F. KOKSMA: Diophantische Approximationen (1936). Ergebnisse der Mathematik, Band **4**, Julius Springer, 157 p., voir p. 95.

Dans la démonstration nous allons introduire un entier $k \geqq 2$. Nous désignerons, quels que soient les nombres réels α et β, la quantité $D\left(\frac{\alpha}{k}, \frac{\beta}{k}\right)$ par $\triangle(\alpha, \beta)$. D'autre part nous désignerons par $N_2(\alpha, \beta)$, $D_2(\alpha, \beta)$, $\triangle_2(\alpha, \beta)$ les quantités $N(\alpha, \beta)$, $D(\alpha, \beta)$, $\triangle(\alpha, \beta)$ relatives à la suite U_2. La démonstration se fera en six étapes.

1⁰) **Identité de Vinogradow** [3]. *Soit $k \geqq 2$ entier et supposons les n nombres $k u_1, \ldots, k u_n$ entiers. Pour tout entier t on a*

$$\sum_{h=0}^{k-1} (\triangle(h, h+t))^2 = \sum_{l=0}^{\tau-1} \triangle_2(-l, l+1) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

où τ désigne la valeur absolue de la différence entre t et le multiple entier de k le plus voisin de t, et où par convention le symbole $\sum_{l=0}^{-1}$ représente 0.

Remarque: Si k augmente indéfiniment, l'identité devient

$$\int_0^1 (D(\alpha, \alpha+\lambda))^2 \, d\alpha = \int_0^\mu D_2(-\alpha, \alpha) \, d\alpha$$

où μ est la valeur absolue de la différence entre λ et l'entier le plus voisin de λ. Cette relation est valable pour toute suite U et tout nombre λ. Comme nous voulons donner une démonstration élémentaire nous n'utiliserons pas cette relation entre intégrales.

L'identité (3) est vérifiée pour $t = 0$, les deux membres de l'identité étant nuls. En vertu de la périodicité en λ de $D(\alpha, \alpha+\lambda)$ le premier membre de (3) est une fonction périodique en t de période k; le deuxième membre de (3) est aussi périodique en t avec la période k, car il en est ainsi de τ. Il suffit donc de démontrer l'idendité pour $0 < t < k$.

Posons $E(h) = \triangle(h, h+1)$, alors en vertu de la propriété additive de la fonction $\triangle$ on a pour tout entier z

$$\sum_{h=0}^{k-1} E(h+z) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Le nombre des u_ν avec

$$u_\nu \equiv \frac{h}{k} \ (mod\ 1)$$

est aussi le nombre des u_ν vérifiant les inégalités:

$$0 \leqq u_\nu - \frac{h}{k} - \left[u_\nu - \frac{h}{k}\right] < \frac{1}{k}$$

c'est-à-dire $N\left(\frac{h}{k}, \frac{h+1}{k}\right) = \left(\frac{1}{k} + E(h)\right) n.$

---

[3] M. Vinogradow a démontré cette identité dans le cas particulier où $k = n$.

Posons $E_2(h) = \triangle_2(h, h+1)$. Si y et l sont des entiers quelconques, le nombre des éléments $u_\nu - u_\varrho$ du système U_2 avec

$$u_\nu - u_\varrho \equiv \frac{y-l}{k} \ (mod\ 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

est donc $n^2\left(\frac{1}{k} + E_2(y-l)\right)$. D'autre part le nombre des u_ν avec

$$u_\nu \equiv \frac{y+h}{k} \ (mod\ 1)$$

est $n\left(\frac{1}{k} + E(y+h)\right)$ et à chacun de ces u_ν correspondent $n\left(\frac{1}{k} + E(l+h)\right)$ nombres u_ϱ avec

$$u_\varrho \equiv \frac{l+h}{k} \ (mod\ 1).$$

Le nombre des éléments $u_\nu - u_\varrho$ du système U_2 avec (5) est donc aussi:

$$\sum_{h=0}^{k-1} \left\{ n\left(\frac{1}{k} + E(y+h)\right) . \, n\left(\frac{1}{k} + E(l+h)\right) \right\}.$$

En vertu de la relation (4) appliquée avec $z = y$ et avec $z = l$, cette expression peut s'écrire:

$$n^2 \left\{ \frac{1}{k} + \sum_{h=0}^{k-1} (E(y+h)\, E(l+h)) \right\},$$

et on a donc:

$$E_2(y-l) = \sum_{h=0}^{k-1} (E(y+h)\, E(l+h)). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

La propriété additive de la fonction $\triangle_2$ permet d'écrire pour tout entier $t \geqq 1$:

$$\sum_{y=0}^{t-1} \triangle_2(y-l, y-l+1) = \triangle_2(-l, t-l),$$

d'où en vertu de (6)

$$\triangle_2(-l, t-l) = \sum_{y=0}^{t-1} E_2(y-l) = \sum_{y=0}^{t-1} \sum_{h=0}^{k-1} (E(y+h)\, E(l+h))$$

et de ce que

$$\sum_{l=0}^{t-1} \triangle_2(-l, l+1) - \sum_{l=0}^{t-1} \triangle_2(-l, t-l) = \sum_{l=0}^{t-1} \triangle_2(t-l, l+1)$$

$$= \sum_{z=0}^{t-1} \triangle_2(z+1, t-z) = -\sum_{z=0}^{t-1} \triangle_2(t-z, z+1) = 0,$$

on a

$$\sum_{l=0}^{t-1} \triangle_2(-l, l+1) = \sum_{l=0}^{t-1} \triangle_2(-l, t-l) = \sum_{l=0}^{t-1} \sum_{y=0}^{t-1} \sum_{h=0}^{k-1} (E(y+h)\,E(l+h))$$

$$= \sum_{h=0}^{k-1} \left(\sum_{y=0}^{t-1} E(y+h) \right)^2 = \sum_{h=0}^{k-1} (\triangle(h, h+t))^2.$$

Enfin remarquons que $\triangle_2(-l', l'+1) = -\triangle_2(-l, l+1)$ lorsque $l + l' = k-1$. Il en résulte que

$$\sum_{l=0}^{t-1} \triangle_2(-l, l+1) = \sum_{l=0}^{\tau-1} \triangle_2(-l, l+1)$$

et l'identité (3) est démontrée.

2⁰) *Soit $k \geqq 2$ entier et supposons les nombres $k u_\nu$ $(\nu = 1, \ldots, n)$ entiers. Pour tout entier r et tout entier $q > 0$ on a l'inégalité:*

$$\sum_{u=r}^{r+k-1} \sum_{v=u-q}^{u+q} (\triangle(u, v))^2 \leqq q(q+1)\,D(U_2). \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Posons $u = v + w$; comme $\triangle(u, v) = -\triangle(v, u)$ le premier membre de (7) peut s'écrire

$$\sum_{w=-q}^{q} \sum_{v=r-w}^{r-w+k-1} (\triangle(v, v+w))^2.$$

En vertu de la périodicité en α de $\triangle(\alpha, \alpha+\lambda)$ on a

$$\sum_{v=r-w}^{r-w+k-1} (\triangle(v, v+w))^2 = \sum_{v=0}^{k-1} (\triangle(v, v+w))^2.$$

L'idendité (3) appliquée avec $t = w$ donne par conséquent:

$$\sum_{u=r}^{r+k-1} \sum_{v=u-q}^{u+q} (\triangle(u, v))^2 = \sum_{w=-q}^{q} \sum_{l=0}^{\omega-1} \triangle_2(-l, l+1),$$

où ω est la valeur absolue de la différence entre w et le multiple de k le plus voisin de w. Le deuxième membre est en valeur absolue inférieur à

$$D(U_2) \sum_{w=-q}^{q} \omega = 2\,D(U_2) \sum_{w=1}^{q} \omega \leqq q(q+1)\,D(U_2).$$

3⁰) *Soit $k \geqq 2$ entier et supposons les nombres $k u_\nu$ $(\nu = 1, \ldots, n)$ entiers. Soient r et s des entiers arbitraires avec $|s-r| < k$, alors*

$$|\triangle(r, s)| \leqq Max \left\{ 4\,\frac{|s-r|}{k},\ 2^4 \sqrt{D(U_2)} \right\}. \quad . \quad . \quad . \quad . \qquad (8)$$

On peut supposer $s \neq r$, car pour $s = r$ le premier membre de (8) s'annule. De même nous pouvons supposer

$$\triangle(r, s)| > 4\,\frac{|s-r|}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Soit j l'intervalle fermé d'extrémités r et s. Pour chaque entier u de j, l'une au moins des quatre inégalités suivantes est vérifiée:

$$\triangle(u, s) \geqq \tfrac{1}{2} \ \triangle(r, s)| \quad \ldots \ldots \ldots \quad (10)$$

$$\triangle(u, s) \leqq -\tfrac{1}{2} \,|\, \triangle(r, s) \quad \ldots \ldots \ldots \quad (11)$$

$$\triangle(r, u) \geqq \tfrac{1}{2} \,|\, \triangle(r, s)$$

$$\triangle(r, u) \leqq -\tfrac{1}{2} \,|\, \triangle(r, s)\,|.$$

En effet dans le cas contraire on aurait

$$|\triangle(r, u)| + |\triangle(u, s)| < |\triangle(r, s)|.$$

Comme j contient $|s-r|+1$ entiers, l'une au moins de ces quatre inégalités est vérifiée par au moins $\frac{1}{4}(|s-r|+1)$ valeurs de u. Puisque $\triangle(\alpha, \beta) = -\triangle(\beta, \alpha)$, on peut même supposer que (10) ou (11) soit vérifiée par au moins $\frac{1}{4}(|s-r|+1)$ entiers u de j, sinon on changerait r en s et s en r. Lorsqu'il en est ainsi pour l'inégalité (10) respectivement (11), nous dirons que nous sommes dans le cas (10) respectivement (11).

Nous désignerons par $j^\star$ l'intervalle fermé de longueur $|s-r|$, d'extrémité s et situé à droite de s dans le cas (10) et à gauche de s dans le cas (11).

Soient h et w des entiers avec

$$h \leqq w \leqq h + |s-r|, \text{ alors } N\left(\frac{h}{k}, \frac{w}{k}\right) \geqq 0$$

donc

$$\triangle(h, w) \geqq -\frac{w-h}{k} \geqq -\frac{|s-r|}{k}$$

et d'après (9)

$$\triangle(h, w) > -\tfrac{1}{4}\,|\triangle(r, s)|. \quad \ldots \ldots \ldots \quad (12)$$

Soit v un entier de $j^\star$. Dans le cas (10) on a $s \leqq v \leqq s + |s-r|$; l'inégalité (12) avec $h = s$, $w = v$ donne

$$\triangle(s, v) > -\tfrac{1}{4}\,|\triangle(r, s)|.$$

Dans le cas (11) on a $v \leqq s \leqq v + |s-r|$; l'inégalité (12) avec $h = v$, $w = s$ donne

$$\triangle(v, s) > -\tfrac{1}{4}\,|\triangle(r, s)|.$$

Il y a donc au moins $\frac{1}{4}(|s-r|+1)$ entiers u de j tels que pour tout v de $j^\star$ on ait (en vertu de la propriété additive de la fonction $\triangle$) dans le cas (10):

$$\triangle(u, v) = \triangle(u, s) + \triangle(s, v) > (\tfrac{1}{2} - \tfrac{1}{4})\,|\triangle(r, s)|,$$

et dans le cas (11):

$$\triangle(u, v) = \triangle(u, s) - \triangle(v, s) < (-\tfrac{1}{2} + \tfrac{1}{4})\,|\triangle(r, s)|.$$

On a donc pour au moins $\frac{1}{4}(|s-r|+1)$ entiers u de j et pour $|s-r|+1$ entiers v de $j^\star$ l'inégalité:

$$|\triangle(u,v)| > \tfrac{1}{4}\,|\triangle(r,s)|.$$

L'inégalité (7) peut être appliquée avec $q = 2\,|s-r|$, et comme

$$|u-v| \leqq 2\,|s-r|,$$

elle donne:

$$\tfrac{1}{4}(|s-r|+1)(|s-r|+1)\,\tfrac{1}{16}\,|\triangle(r,s)|^2 < 2\,|s-r|\,(2\,|s-r|+1)\,D(U_2)$$

d'où

$$|\triangle(r,s)| < 16\sqrt{D(U_2)}$$

et l'inégalité (8) est démontrée.

4⁰) *Soit $k \geqq 2$ entier et supposons les nombres $k u_\nu$ $(\nu = 1, \ldots, n)$ entiers. Pour tout nombre réel $\sigma > 0$, tout entier $m \geqq 1$ et pour chaque couple d'entiers r et s avec $|s-r| < k$ on a l'inégalité:*

$$|\triangle(r,s)| \leqq \mathrm{Max}\left\{\varphi(m,\sigma)\left(\frac{|s-r|}{k}\,(D(U_2))^{m-1}\right)^{\frac{1}{2m-1}};\ 2^{m+3}(1-2^{-\sigma})^{-m+1}\sqrt{D(U_2)}\right\} \qquad (13)$$

où

$$\varphi(m,\sigma) = \left\{2^{m^2+(2\sigma+1)m-2\sigma}(1-2^{-\sigma})^{-(m-1)^2}\right\}^{\frac{1}{2m-1}}.$$

L'inégalité (13) est vérifiée pour $m = 1$ car elle se réduit alors à (8). Nous pouvons donc supposer que $m \geqq 2$ et que l'inégalité (13) soit déjà démontrée pour $m-1$ au lieu de m.

On peut admettre que $r \neq s$, le premier membre de (13) étant nul pour $r = s$. De plus nous pouvons supposer que

$$|\triangle(r,s)| > 2^{m+3}(1-2^{-\sigma})^{-m+1}\sqrt{D(U_2)}. \qquad (14)$$

Enfin nous pouvons supposer que le nombre d'entiers u de l'intervalle j avec

$$|\triangle(u,s)| \geqq \tfrac{1}{2}\,|\triangle(r,s)| \qquad (15)$$

est au moins $\frac{1}{2}(|s-r|+1)$; on échangerait dans le cas contraire r et s.

Soient alors v les entiers de j pour lesquels on a

$$|s-v| \leqq k\left\{\frac{(1-2^{-\sigma})}{2\,\varphi(m-1,\sigma)}\,|\triangle(r,s)|\right\}^{2m-3}(D(U_2))^{-m+2}. \qquad (16)$$

Nous désignerons le second membre de cette inégalité par g.

En appliquant à un entier v vérifiant (16) l'inégalité (13) avec $m-1$ à la place de m et v à la place de r, il vient:

$$|\triangle(v,s)| < \mathrm{Max}\left\{\varphi(m-1,\sigma)\left(\frac{|s-v|}{k}\,(D(U_2))^{m-2}\right)^{\frac{1}{2m-3}};\ 2^{m+2}(1-2^{-\sigma})^{-m+2}\sqrt{D(U_2)}\right\}$$

ce qui d'après (16) donne:

$$|\triangle (v, s)| < Max\{\tfrac{1}{2}(1-2^{-\sigma})|\triangle (r, s) \;;\; 2^{m+2}(1-2^{-\sigma})^{-m+2}\sqrt{D(U_2)}\}$$

et enfin en tenant compte de (14)

$$|\triangle (v, s)| < \tfrac{1}{2}(1-2^{-\sigma})|\triangle (r, s)|. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

Comme $\frac{1}{2}(1-2^{-\sigma}) < \frac{1}{2}$ parce que $\sigma > 0$, l'inégalité (17) n'est pas vérifiée pour $v = r$, donc $v = r$ ne satisfait pas (16). Il en résulte que $g < |s-r|$ et le nombre d'entiers v qui entrent en considération est le plus grand entier ne dépassant pas $g + 1$, donc est supérieur à g.

Par suite pour au moins $\frac{1}{2}(|s-r| + 1)$ entiers u de j et au moins g entiers v de j on a les inégalités (15) et (17), et par conséquent

$$\triangle (u, v)| \geqq |\triangle (u, s)| - |\triangle (v, s)| > (\tfrac{1}{2} - \tfrac{1}{2}(1-2^{-\sigma}))|\triangle (r, s)|$$

c'est-à-dire

$$|\triangle (u, v)| > 2^{-\sigma-1}|\triangle (r, s)|.$$

L'inégalité (7) peut être appliquée avec $q = |s-r|$, et comme

$$|u-v| \leqq |s-r|,$$

elle donne:

$$\tfrac{1}{2}(|s-r| + 1)\, g\, 2^{-2(\sigma+1)}|\triangle (r, s)|^2 < |s-r|(|s-r| + 1)\, D(U_2)$$

c'est-à-dire

$$|\triangle (r, s)|^{2m-1} < 2^{2\sigma+3+2m-3}(1-2^{-\sigma})^{-2m+3}(\varphi(m-1, \sigma))^{2m-3}\frac{|s-r|}{k}(D(U_2))^{m-1}.$$

En tenant compte de la valeur de $\varphi(m-1, \sigma)$ on voit que l'on obtient:

$$\triangle (r, s)| < \varphi(m, \sigma)\left(\frac{|s-r|}{k}(D(U_2))^{m-1}\right)^{\frac{1}{2m-1}}$$

et l'inégalité (13) est démontrée.

5°) *Démonstration des inégalités (1) et (2) lorsque $k \geqq 2$ est entier et que les nombres $k u_\nu$ $(\nu = 1, \ldots, n)$ sont entiers.*

Posons $D(U_2) = 2^{-d}$, alors $d = \dfrac{|\log D(U_2)|}{\log 2}$ est un nombre positif, car $D(U_2) \leqq 1$.

Nous appliquerons la formule (13) où nous avons posé $\sigma = \dfrac{m}{2}$ et remplacé $|s-r|$ par k. La formule (13) devient alors:

$$D(U) \leqq Max \left\{ \begin{array}{l} 2^{\frac{1}{2(2m-1)}} \left(1-2^{-\frac{m}{2}}\right)^{-\frac{(m-1)^2}{2m-1}} 2^{m+\frac{1}{2}+\frac{d}{2(2m-1)}} \sqrt{D(U_2)}; \\ \left(1-2^{-\frac{m}{2}}\right)^{-m+1} 2^{m+3} \sqrt{D(U_2)} \end{array} \right\} . \qquad (18)$$

On vérifie que, quel que soit l'entier $m \geqq 1$, on a

$$2^{\frac{1}{2(2m-1)}} \left(1-2^{-\frac{m}{2}}\right)^{-\frac{(m-1)^2}{2m-1}} < 2$$

et

$$\left(1-2^{-\frac{m}{2}}\right)^{-m+1} < 2^{\frac{3}{2}} .$$

Il suffit pour cela de constater que la dérivée par rapport à m de chacune de ces expressions est positive pour $1 \leqq m \leqq 3$ et négative pour $m \geqq 4$. La substitution des valeurs $m = 3$ et $m = 4$ fournit les inégalités écrites.

L'inégalité (18) devient alors:

$$D(U) < Max \left\{ 2^{m+\frac{3}{2}+\frac{d}{2(2m-1)}} ; 2^{m+\frac{9}{2}} \right\} \sqrt{D(U_2)} .$$

Supposons d'abord $d \geqq 36$ et prenons pour m l'entier défini par les inégalités

$$1 + \sqrt{d} \leqq 2m < 3 + \sqrt{d} .$$

Alors

$$m + \frac{3}{2} + \frac{d}{2(2m-1)} < 3 + \sqrt{d} \text{ et } m + \frac{9}{2} < 6 + \frac{1}{2}\sqrt{d} .$$

Or pour $d \geqq 36$ on a $3 + \sqrt{d} \geqq 6 + \frac{1}{2}\sqrt{d}$, donc dans ce cas on a

$$D(U) < 2^{3+\sqrt{d}} \sqrt{D(U_2)} .$$

Supposons maintenant $d < 36$ et prenons pour m l'entier défini par les inégalités

$$1 + \frac{d}{6} \leqq 2m < 3 + \frac{d}{6} .$$

Alors

$$m + \frac{3}{2} + \frac{d}{2(2m-1)} < 6 + \frac{d}{12} \text{ et } m + \frac{9}{2} < 6 + \frac{d}{12} .$$

On a donc dans ce cas

$$D(U) < 2^{6+\frac{d}{12}} \sqrt{D(U_2)} .$$

On voit facilement que pour $\frac{72}{5} < d < 36$ on a $6 + \frac{d}{12} < \frac{7}{2} + \sqrt{d}$. Si d est $\leqq \frac{72}{5}$, on a $\frac{7}{2} + \sqrt{d} - \frac{1}{2} d > 0$, de sorte que l'on a:

$$D(U) < 2^{\frac{7}{2} + \sqrt{d} - \frac{1}{2} d} = 2^{\frac{7}{2} + \sqrt{d}} \sqrt{D(U_2)}$$

inégalité évidente, car le second membre est supérieur à 1. On a donc pour toutes les valeurs de d

$$D(U) < 2^{\frac{7}{2} + \sqrt{d}} \sqrt{D(U_2)}$$

et l'inégalité (1) est démontrée pour la suite U.

Pour démontrer l'inégalité (2) nous écrivons, $\varepsilon > 0$ étant un nombre arbitraire:

$$D(U) < 2^{\frac{7}{2} + \sqrt{d} - \frac{d}{2}} = 2^{\frac{7}{2} + \sqrt{d} - \varepsilon d - \left(\frac{1}{2} - \varepsilon\right) d} = 2^{\frac{7}{2} + \sqrt{d} - \varepsilon d} (D(U_2))^{\frac{1}{2} - \varepsilon}.$$

Or $\frac{7}{2} + \sqrt{d} - \varepsilon d$ prend sa plus grande valeur pour $\sqrt{d} = \frac{1}{2\varepsilon}$ et par suite:

$$D(U) < 2^{\frac{7}{2} + \frac{1}{2\varepsilon} - \frac{1}{4\varepsilon}} (D(U_2))^{\frac{1}{2} - \varepsilon} = 2^{\frac{7}{2} + \frac{1}{4\varepsilon}} (D(U_2))^{\frac{1}{2} - \varepsilon}$$

et l'inégalité (2) est démontrée.

L'inégalité (2) est équivalente à (1) car on retrouve (1) en prenant dans (2) pour ε la valeur particulière $\frac{1}{2\sqrt{d}}$.

6^0) *U étant un système de n nombres réels $u_1, \ldots, u_n$ quelconques et $U^\star$ un système de n nombres réels $u_1^*, \ldots, u_n^*$ tels que $|u_\nu - u_\nu^*| \leqq \eta$ $(\nu = 1, \ldots, n)$, où η est une constante indépendante de ν, on a l'inégalité:*

$$|D(U) - D(U^\star)| \leqq 2\eta. \quad \ldots \ldots \quad (19)$$

Désignons par $N^\star(\alpha, \alpha + \lambda)$ le nombre des éléments u_ν^* de $U^\star$ avec

$$0 \leqq u_\nu^* - \alpha - [u_\nu^* - \alpha] < \lambda - [\lambda].$$

Si $0 \leqq \lambda < 1 - 2\eta$, on a

$$N(\alpha, \alpha + \lambda) \leqq N^*(\alpha - \eta, \alpha + \lambda + \eta) \leqq n(\lambda + 2\eta + D(U^*)),$$

donc

$$D(\alpha, \alpha + \lambda) \leqq 2\eta + D(U^*).$$

Si $1-2\eta \leqq \lambda < 1$, on a

$$N(\alpha, \alpha+\lambda) \leqq n + N^*(\alpha-\eta, \alpha+\lambda-1+\eta) \leqq n\,(1+\lambda-1+2\eta+D(U^*)),$$

donc

$$D(\alpha, \alpha+\lambda) \leqq 2\eta + D(U^*).$$

En vertu de la périodicité en λ de $D(\alpha, \alpha+\lambda)$ on a par conséquent pour tout couple α et λ

$$D(\alpha, \alpha+\lambda) \leqq 2\eta + D(U^*),$$

c'est-à-dire on a

$$D(U) \leqq 2\eta + D(U^*).$$

En échangeant les rôles des suites U et U^* on obtient de même:

$$D(U^*) \leqq 2\eta + D(U)$$

et l'inégalité (19) est démontrée.

Fin de la démonstration.

Soit U un système de n nombres réels $u_1, \ldots, u_n$ quelconques et soit k un entier arbitraire. Définissons un système U^* de n nombres $u_1^* \ldots, u_n^*$ par la propriété que $k u_\nu^*$ est l'entier le plus voisin de $k u_\nu$ pour $\nu = 1, \ldots, n$. Ainsi $k u_\nu^*$ est entier et $|u_\nu - u_\nu^*| \leqq \frac{1}{2k}$.

On a démontré pour le système U^* l'inégalité (2) quel que soit $\varepsilon > 0$, c'est-à-dire on a

$$D(U^*) < 2^{\frac{7}{2}+\frac{1}{4\varepsilon}} (D(U_2^*))^{\frac{1}{2}-\varepsilon}.$$

L'inégalité (19) peut être appliquée aux systèmes U et U^* avec $\eta = \frac{1}{2k}$ et aux systèmes U_2 et U_2^* avec $\eta = \frac{1}{k}$. Par suite

$$D(U) < \frac{1}{k} + 2^{\frac{7}{2}+\frac{1}{4\varepsilon}} \left(\frac{2}{k} + D(U_2)\right)^{\frac{1}{2}-\varepsilon}.$$

Cette inégalité a lieu pour tout k; en faisant augmenter indéfiniment k il vient

$$D(U) \leqq 2^{\frac{7}{2}+\frac{1}{4\varepsilon}} (D(U_2))^{\frac{1}{2}-\varepsilon}.$$

On peut même remplacer le signe $\leqq$ par $<$ car on aurait pu prendre au lieu de $\frac{7}{2}$ dans l'exposant une constante absolue un peu plus petite. L'inégalité (2), et par suite aussi l'inégalité (1), sont donc démontrées pour la suite U.

Mathematics. — *On a theorem in the geometry of numbers and a property of mass distributions in n-dimensional space.* By CORNELIS VISSER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of May 20, 1939.)

§ 1. *Introduction.*

Let R_n be the n-dimensional Cartesian space. By $x + y$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are points in R_n, we denote the point $(x_1 + y_1, \dots, x_n + y_n)$ and by $E + y$, where E is a set and y a point in R_n, the set of all points $x + y$ with x belonging to E. $E + y$ will be called a translation of E.

We shall indicate by H_N^a the cube

$$a_j - N \leqq x_j \leqq a_j + N \qquad (j = 1, \dots, n).$$

Let $\xi^{(1)}, \xi^{(2)}, \dots$ be a sequence of points; the number

$$\limsup_{a \text{ arbitrary},\, N \to \infty} \frac{\text{number of } \xi^{(i)} \text{ in } H_N^a}{(2N)^n}$$

will be called the *upper density* of the sequence $\xi^{(i)}$.

The following theorem, which is a generalization of a well-known theorem of BLICHFELDT [1], is due to JESSEN [2].

Theorem 1. *Let $\xi^{(i)}$ be a sequence of points with upper density $> \lambda$. Let E be a measurable set of finite* LEBESQUE *measure $m(E)$. Then there exists a translation of E which contains more than $\lambda\, m(E)$ of the points $\xi^{(i)}$.*

In this note we shall prove a further generalization of BLICHFELDT's theorem, which includes Theorem 1 as a special case.

Instead of a sequence of points we consider an arbitrary mass distribution. By this we mean a non-negative set function $\sigma(E)$ which is completely additive over a field of sets that includes at least all BOREL sets. The sets of this field will be referred to as the sets which are measurable with respect to σ. Further we shall call $\sigma(E)$ the amount of mass contained in E. It is understood that σ has a finite value for all bounded sets.

---

[1] H. F. BLICHFELDT: A new principle in the geometry of numbers, with some applications. Trans. Am. Math. Soc. **15**, 227—235 (1914).

[2] See the "Zusatz bei der Korrektur" in WERNER FENCHEL: Verallgemeinerungen einiger Sätze aus der Geometrie der Zahlen. Acta Arithmetica **2**, 230—241 (1937).

The *upper density* of σ is the number

$$\limsup_{a \text{ arbitrary}, N \to \infty} \frac{\sigma(H_N^a)}{(2N)^n}.$$

Theorem 2. *Let σ be a mass distribution with upper density $> \lambda$. Let E be an arbitrary set with finite interior* LEBESQUE *measure $m_i(E)$. Then there exists a translation of E which contains an amount of mass greater than $\lambda\, m_i(E)$.*

Consider the sequence $\xi^{(i)}$ of Theorem 1. Define $\sigma(E)$ for all subsets E of R_n as the number of points $\xi^{(i)}$ belonging to E. Then σ is a mass distribution as considered in Theorem 2 and by application of this theorem we get Theorem 1.

Thus Theorem 1 is a special case of Theorem 2. We shall, however, include a direct proof of Theorem 1 in § 2.

§ 2. *Proof of Theorem 1.*

We shall make use of the following

Lemma. *Let $F_1, F_2, \ldots$ be a finite or infinite sequence of measurable sets all lying in a set S of finite positive measure. Then there exists a point such that the number of sets F_i to which it belongs is at least* $\dfrac{\sum_i m(F_i)}{m(S)}$.

To see this, consider the characteristic functions $f_i(x)$ of the sets F_i. Since

$$\sum_i m(F_i) = \sum_i \int_S f_i(x)\, dx = \int_S \sum_i f_i(x)\, dx,$$

it is impossible that for all x in S $\sum_i f_i(x)$ is smaller than $\dfrac{\sum_i m(F_i)}{m(S)}$.

Hence there is a point x with $\sum_i f_i(x) \geqq \dfrac{\sum_i m(F_i)}{m(S)}$, which means that the number of sets F_i to which x belongs is at least $\dfrac{\sum_i m(F_i)}{m(S)}$.

We now prove Theorem 1 in the following manner.

It is sufficient to consider the case that E is bounded. We may further suppose that E contains the origin. Let d be the diameter of E. There exists a cube H_N^a such that

$$\frac{\text{number of } \xi^{(i)} \text{ in } H_N^a}{(2N+2d)^n} > \lambda.$$

Now

$$\sum_i m\left(E+a\,.\,H^{\xi^{(i)}}_{N+d}\right) \geqq \sum_{\xi^{(i)}\text{ in }H^a_N} m\left(E+a\,.\,H^{\xi^{(i)}}_{N+d}\right)$$

$$= \sum_{\xi^{(i)}\text{ in }H^a_N} m(E+a) = m(E)\,.\,\text{number of } \xi^{(i)} \text{ in } H^a_N$$

$$> \lambda m(E)(2N+2d)^n.$$

Put

$$F_i = E + a\,.\,H^{\xi^{(i)}}_{N+d} - \xi^{(i)}.$$

All F_i are contained in H^0_{N+d} and

$$\sum_i m(F_i) = \sum_i m\left(E+a\,.\,H^{\xi^{(i)}}_{N+d}\right) > \lambda m(E)(2N+2d)^n.$$

Hence, by the lemma, there exists a point x which belongs to more than $\dfrac{\lambda\, m(E)(2N+2d)^n}{m(H^0_{N+d})} = \lambda m(E)$ of the sets F_i, say $F_{i_1}, \ldots, F_{i_p}$. But then the points $\xi^{(i_1)}, \ldots, \xi^{(i_p)}$ belong to $E+a-x$, which proves the theorem.

§ 3. *Proof of Theorem 2.*

Since the interior measure of a set is the least upper bound of the measures of its closed bounded subsets it is sufficient to consider the case of a closed bounded set E. We may further assume that E contains the origin. Let d denote its diameter.

There exists a cube H^a_N such that

$$\frac{\sigma(H^a_N)}{(2N+2d)^n} > \lambda.$$

The function $m(E+a\,.\,H^{\xi}_{N+d})$ is continuous in ξ and hence measurable with respect to σ.

We have

$$\int_{R_n} m(E+a\,.\,H^{\xi}_{N+d})\,\sigma(d\xi) \geqq \int_{H^a_N} m(E+a\,.\,H^{\xi}_{N+d})\,\sigma(d\xi)$$

$$= \int_{H^a_N} m(E+a)\,\sigma(d\xi) = m(E)\,\sigma(H^a_N) > \lambda m(E)(2N+2d)^n.$$

Now consider the product P of the cube H_{N+d}^0 and R_n, i. e. the space whose points are the pairs (x, ξ), where x is a point of H_{N+d}^0, ξ a point of R_n. We denote by μ the product measure of the measure m in H_N^0 and the measure σ in R_n.[1]) Let A be the subset of P consisting of all pairs (x, ξ) for which $x+\xi$ belongs to $E+a\,.\,H_{N+d}^{\xi}$. It is easy to see that A is a closed set. Hence A is measurable with respect to μ; further we may apply FUBINI's theorem.

Let A_x be the set of all points ξ such that (x, ξ) belongs to A, A_ξ the set of all points x such that (x, ξ) belongs to A. Then

$$\begin{aligned}\mu(A) &= \int_{R_n} m(A_\xi)\,\sigma(d\xi) \\ &= \int_{R_n} m(E+a\,.\,H_{N+d}^{\check{\xi}}-\xi)\sigma(d\xi) \\ &= \int_{R_n} m(E+a\,.\,H_{N+d}^{\check{\xi}})\,\sigma(d\xi) > \lambda\, m(E)\,(2N+2d)^n.\end{aligned}$$

But also

$$\mu(A) = \int_{H_{N+d}^0} \sigma(A_x)\, m(dx).$$

Hence there must exist a point x in H_{N+d}^0 such that

$$\sigma(A_x) > \frac{\lambda\, m(E)\,(2N+2d)^n}{m(H_{N+d}^0)} = \lambda\, m(E).$$

Now, by the definition of A_x, any point $x+\xi$ with ξ in A_x belongs to $E+a$. Hence

$$A_x + x \subset E + a.$$

Therefore

$$A_x \subset E + a - x, \quad \text{and} \quad \sigma(A_x) > \lambda\, m(E),$$

which proves Theorem 2.

---

[1]) See S. SAKS: Theory of the Integral, second revised edition (Warszawa—Lwów, 1937).

Mathematics. — *On the* DIRICHLET *Problem and the Method of Sweeping-out.* By A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of May 20, 1939.)

§ 1. As is well known, there are several methods by which a harmonic function can be constructed for a given bounded open set with continuous boundary values, in such a way that, if the DIRICHLET problem has a solution in the given case, the harmonic function coincides with that solution. All these methods lead to the same function $F_W(P)$ which is the solution of the following problem:

Let Ω be a bounded open set, Σ its boundary and $f(Q)$ a continuous function on Σ. Then a function $F(P)$, harmonic in Ω, is to be determined in such a way that

$$\lim_{P \to Q} F(P) = f(Q)$$

with the possible exception of a set of capacity zero.

This problem has the unique solution $F_W(P)$ which can be proved to be of the form:

$$F_W(P) = \int_{\Sigma} f(Q)\, d\mu^P(e_Q), \quad \dots \dots \dots \quad (1)$$

where $\mu^P(e)$ means a positive set function, defined for all measurable (in the sense of BOREL) subsets of Σ and depending on the point P in Ω.

By these methods a harmonic function is assigned to every bounded open set with continuous boundary values, which harmonic function coincides with a given harmonic function for special, geometrically characterizable, open sets (sets for which the DIRICHLET problem has a solution). The question then arises whether this function should be determined solely by this fact together with the linearity of the problem, without making a single further condition on the behaviour on the boundary. This question shows a large analogy with a well-known problem in the theory of set functions. We refer to the problem where an absolutely continuous, completely additive set function is to be constructed when its values for special sets are given (e. g. intervals).

So we have the following problem:

To each bounded open set Ω with continuous boundary values $f(Q)$ a function $F(P)$, harmonic in Ω, be assigned in such a way that:

1. *F coincides with the solution of the* DIRICHLET *problem, when existing;*

2. *F is a bounded linear functional in the space (C) of the continuous functions $f(Q)$ (as distance of f_1 and f_2 be taken the last upper bound of $|f_1 - f_2|$);*

3. *If $\Omega_1 + \Sigma_1 \subset \Omega$, F corresponding to Ω, then $F(P)$ also corresponds to $\Omega' \equiv \Omega - (\Omega_1 + \Sigma_1)$ for boundary values F on Σ_1 and f on Σ.*

As $F_w(P)$ satisfies these conditions, a function F certainly exists. It is to be proved that this is the only solution.

In the following paragraphs we will prove this theorem in a special case.

§ 2. *Theorem. Every function which satisfies the conditions of § 1 can be written in the form of a* STIELTJES—RADON-*integral extended over Σ:*

$$F(P) = \int_{\Sigma} f(Q)\, du^P(e_Q), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where $u^P(e)$ means a completely additive positive set function, defined, independently of $f(Q)$, for all measurable (B) subsets e of Σ and depending on the point $P \subset \Omega$ as parameter; $u^P(\Sigma) = 1$.[1]

Proof. Be (M_B) the space of the bounded functions $g(Q)$ where $g(Q)$ is measurable (B) (i.e. a function belonging to a class of BAIRE). The least upper bound of $g_1 - g_2$ again being taken as distance of g_1 and g_2, (M_B) contains (C). It can then be proved that (for every P in Ω) a linear bounded functional exists in (M_B) coinciding with F on (C)[2]; we denote this functional still by $F(P)$.

Now let e be a subset of Σ belonging to the class (B). As the characteristic function $\varphi(Q)$ of e ($\varphi(Q) = 1$ for $Q \subset e$, $\varphi(Q) = 0$ for $Q \subset \Sigma - e$) is measurable (B), $\varphi(Q) \subset (M_B)$. For any fixed $P \subset \Omega$ we then put

$$u^P(e) = F[\varphi(Q)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

With this definition the relation (2) can easily be proved[3]. Remark that only the second condition is used.

It is possible for any fixed $P \subset \Omega$ to extend the range of definition of $u^P(e)$ to the class M of the so-called u^P-measurable sets in such a way that $u^P(e)$ is still completely additive over this class. The criteria to test whether or not a set e belongs to M are analogous to those in the theory of LEBESQUE measure. We will show that this definition of a

---

[1] It is obvious that several functions u^P would exist, if Fw were not the only solution. We refer to u^P as a distribution of a positive mass 1 on Σ.

[2] See for instance St. BANACH, Théorie des opérations linéaires, Warszawa 1932, Ch. IV.

[3] A. F. MONNA, Het probleem van DIRICHLET, Nieuw Archief voor Wiskunde, XIX, (1938).

u^P-measurable set has a meaning independent of the position of P. First we note that $0 < u^P(e) < 1$ for any set $e \subset \Sigma$ and any $P \subset \Omega$ (MONNA l. c.).

Let us then suppose that the set e_0 is measurable when $P = P_1$; we will show that this is also the case when $P = P_2$ for any $P_2 \neq P_1$. It is obviously possible to construct sets e'_n and e''_n $(n = 1, 2, \ldots)$ respectively open and closed, so that

$$e''_1 \subseteqq \ldots \subseteqq e''_n \subseteqq e''_{n+1} \ldots \subseteqq e_0 \subseteqq \ldots \subseteqq e'_{n+1} \subseteqq e'_n \subseteqq \ldots \subseteqq e'_1,$$

and

$$u^{P_1}(e'_n - e''_n) < \varepsilon_n$$

where $\varepsilon_n > 0$ tends to zero when $n \rightarrow \infty$. Now, according to (3), $u^P(e)$ is the harmonic function corresponding to values 1 on e and 0 on $\Sigma - e$.[1]) As $e'_{n+1} - e''_{n+1} \subseteqq e'_n - e''_n$, it follows that

$$u^P(e'_1 - e''_1) \geqq u^P(e'_2 - e''_2) \geqq \ldots,$$

and as this series converges to zero for $P = P_1$, according to a theorem of HARNACK, it converges in Ω to a harmonic function with values between 0 and 1. Since this function reaches its lower bound in the interior point P_1, it is identically equal to zero. So we have

$$\lim_{n \rightarrow \infty} u^{P_2}(e'_n - e''_n) = 0,$$

which means: e is measurable when $P = P_2$.

§ 3. From (1) and (2) it follows that it is sufficient to show that $u^P(e) = \mu^P(e)$. Our proof of this fact (in a special case) will be based essentially on the following theorem, proved in the theory of sweeping-out [2]):

Let α denote a distribution of positive mass on the bounded open set Ω and be V its potential. Then one and only one distribution β of this mass on Σ exists, such as to make the potential U of β satisfy the relations:

$V > U$ in Ω.

$V = U$ outside Ω and on Σ with the possible exception of a set of capacity zero.

In all regular boundary points of Σ we have then $V = U$. The transformation of the mass α into β is called "sweeping-out".

If for α we take the mass 1 placed in $P \subset \Omega$, by this method a distribution $\mu^P(e)$ on Σ is obtained and it can be proved that it is this distribution which figures in formula (1).

---

[1]) See also: G. C. EVANS. On potentials of positive mass. Transactions of the Amer. Math. Soc. Vol. **38**, 215, (1935).

[2]) M. BRELOT, Fonctions sous-harmoniques et balayage. Académie royale de Belgique, Bulletin de la classe des Sciences XXIV, (1938).

In view of the application of this theorem we derive some properties of the potential of the distribution $u^P(e)$ in a point R:

$$U(P;R)=\int\frac{du^P(e_Q)}{RQ}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We compare this potential with the potential of the mass 1 placed in the fixed point $P \subset \Omega$.

a. R be a point outside Ω.

As function of P, U is the harmonic function in Ω corresponding to the boundary values $\frac{1}{RQ}$. As for these values the DIRICHLET problem has a solution $\frac{1}{PR}$, we have, according to the first condition,

$$U(P;R)=\frac{1}{PR}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$(P \subset \Omega;\ R \text{ outside } \Omega).$$

b. R be a point in Ω.

Keep R fixed and let B_δ be the closed sphere with radius δ and centre R. Consider

$$U^*(P;R)=\frac{1}{PR}-U(P;R)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in $\Omega - B_\delta$ as function of P. Applying the third condition, we see easily that $U^* > 0$. As δ is arbitrary, we have for all P and R in Ω

$$U(P;R)<\frac{1}{PR}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$(P \subset \Omega;\ R \subset \Omega).$$

From these relations it follows that the potential $U(P;R)$ (a function of R) is everywhere bounded for every fixed P in Ω[1]). This is clear in all points at a positive distance from Σ (outside the mass). For points in the neighbourhood of Σ it is proved as follows: denote by $\triangle$ the shortest distance of P to Σ ($\triangle$ is a positive quantity $\neq 0$). Then it follows from (7) that in the neighbourhood of Σ

$$U(P;R)<\frac{1}{\triangle}.$$

Since now the potential is bounded everywhere outside Σ, it is also bounded on Σ.

As a consequence of this property, the distribution u^P bears no mass on sets of capacity zero, i.e. $u^P(e)=0$ if the capacity of e is equal to zero [1]). It follows then from (2) that a change of the values of $f(Q)$ on a set of capacity zero does not change the corresponding harmonic function. In particular, if Σ contains a subset e with capacity zero, such that $\Omega+e$ is still an open set, the points of e may be added to Ω; the harmonic function in $\Omega+e$, with the given boundary values on $\Sigma-e$, coincides in Ω with the harmonic function corresponding to the given values on Σ. Points of e would give rise only to removable singularities. As now every closed set is the sum of a perfect and a denumerable set (i.e. a set of capacity zero), we see that it may be assumed that Σ is a perfect set.

---

[1]) We remark that from this point onward, P denotes a fixed but arbitrary point in Ω; the potentiated point is R.

In that case it can be proved that the potential is continuous in all the points of a subset of Σ, everywhere dense on Σ [2]). The value of U in the continuity points Q_e belonging to the exterior boundary follows from (5):

$$U(P;Q_e)=\frac{1}{PQ_e}.$$

Applying the criterion of BOULIGAND, we see that these points are regular boundary points for Ω.

Remark. If we assume that $u^P(e)=\mu^P(e)$, it is easy to see that from $u^P(e)=0$ it follows that the capacity of e is zero. Indeed, the harmonic function corresponding to values 1 on e and 0 on $\Sigma-e$ then takes in no point of e the value 1, so all the points of e are then irregular. As the set of all irregular points of Ω has a capacity zero, a fortiori the capacity of e is zero.

§ 4. For the application of the theorem of § 3 it is necessary to prove that $U(P,Q)=\frac{1}{PQ}$ in the points Q of Σ with the possible exception of a set of capacity zero. In the following special case this can be done as follows.

Let $m(\varrho)$ be the measure (in the sense of LEBESGUE) of the set of the points which do not belong to Ω and which lie upon the sphere of radius ϱ with the point Q as centre. We will now suppose:

1. *Ω has no interior boundary;*
2. *The boundary points where*

$$\lim \frac{m(\varrho)}{\varrho^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

form a set e of capacity zero.

---

[1]) See for instance C. DE LA VALLÉE POUSSIN, Les nouvelles méthodes de la théorie du potentiel et le problème généralisé de DIRICHLET. Actualités scientifiques et ind. 578 (1937).

[2]) G. C. EVANS, Transactions of the Am. Math. Soc. Vol. 37.
F. VASILESCO, Journal de Math. 1935.

We consider, as function of R,

$$U^*(P,R) = \frac{1}{PR} - \int\limits_{\Sigma} \frac{du^P(e_Q)}{RQ}. \quad \ldots \ldots \ldots \quad (9)$$

A potential of negative mass being subharmonic, this function is subharmonic everywhere with the only exception of the point $R = P$. It follows then that U^* is quasi continuous in every point (except $R = P$), in particular in every point Q of Σ, that is, has the following property [1]).

We call "quasi-neighbourhood" of the point Q a part of the neighbourhood of that point for which

$$\lim_{\varrho \to \varrho} \frac{\alpha(\varrho)}{\varrho^2} = 1,$$

where $\alpha(\varrho)$ means the measure of the section of that part by a sphere with radius ϱ round Q. Then every point Q has a quasi-neighbourhood in which U^* is a continuous function of R.

Be now Q_0 a point of Σ, not belonging to e. From the definition of e it follows then immediately that the part of every neighbourhood of Q_0 outside Ω certainly contains points R_e of the quasi-neighbourhood of that point (if not so, $\frac{m(\varrho)}{\varrho^2} \to 0$ and Q_0 belonged to e). By (5) we have in these points R_e

$$U^*(P; R_e) = 0,$$

and so, U^* being quasi-continuous,

$$U(P; Q_0) = \frac{1}{PQ_0}. \quad \ldots \ldots \ldots \quad (10)$$

Since the points Q where (10) possibly is not satisfied form a set e of capacity zero, it follows from (5), (7) and (10) that we can apply the theorem of § 3, taking for α the mass 1 placed in P with potential $\frac{1}{PR}$. So we have

$$u^P(e) = \mu^P(e)$$

and then

$$F(P) = F_W(P).$$

Remarks.

1. In every irregular point we have

$$\lim_{\varrho \to 0} \frac{m(\varrho)}{\varrho^2} = 0,$$

---

[1]) M. BRELOT, Etude des fonctions sousharmoniques au voisinage d'un point. Actualités scient et ind., Vol. 139 (1934).

and as the irregular points form a set of capacity zero, the second condition imposed on Ω means a restriction on the regular points of Ω. The regular points where (8) is satisfied must form a set of capacity zero.

2. It is possible to reduce the first condition somewhat. Suppose:

1. *The interior boundary* Σ_i *is the sum of a denumerable number of closed sets:* $\Sigma_i = \Sigma_i^{(1)} + \Sigma_i^{(2)} + \dots$;
2. *The points of* Σ_i *are regular for* Ω;
3. *The points of the exterior boundary* Σ_e, *where*

$$\lim \frac{m(\varrho)}{\varrho^2} = 0,$$

form a set of capacity zero.

The proof then is as follows. It means no restrictiction when we suppose that no $\Sigma_i^{(k)}$ and $\Sigma_i^{(l)}$ have points in common. In that case it is possible to construct open sets $\Omega^{(k)}$ (boundary $\Sigma^{(k)}$) so that $\Omega^{(k)} + \Sigma^{(k)}$ $(k = 1, 2, \dots)$ have no points in common and $\Omega^{(k)} + \Sigma^{(k)} \subset \Omega$; $\Sigma_i^{(k)} \subset \Omega^{(k)}$ (remark that $\Sigma_i^{(k)}$ and Σ_e have a positive distance). We may furthermore assume that the DIRICHLET problem for $\Omega^{(k)}$ has a solution for any continuous boundary value. As regularity of a boundary point is a local property, the same is then, according to 2^0, true for the open set $\Omega'^{(k)}$, contained in $\Omega^{(k)}$ and having $\Sigma^{(k)}$ and $\Sigma_i^{(k)}$ as boundary. Applying then the third condition of § 1, we see that $F(P)$ is in $\Omega'^{(k)}$ the solution of the DIRICHLET problem for values $F(P)$ on $\Sigma^{(k)}$ and $f(Q)$ on $\Sigma_i^{(k)}$ $(k = 1, 2, \dots)$. So we have in every point Q_i of Σ_i

$$\lim_{P \to Q_i} F(P) = f(Q_i).$$

As $\Omega^* = \Omega - (\Omega^{(1)} + \Sigma^{(1)}) - (\Omega^{(2)} + \Sigma^{(2)}) - \dots$ has no interior boundary, we may, on account of 3^0 and the third condition first mentioned, in Ω^* apply the result of § 4. So we have in Ω^*

$$F(P) = F_W^*(P),$$

F_W^* taken for the corresponding boundary values. In the regular points of Σ_e we have then

$$\lim_{P \to Q} F(P) = f(Q).$$

This equality holds good in every regular point of Σ. As two functions, both harmonic in a domain, are identical when they assume in all the regular boundary points of Ω the same continuous boundary values ¹), we have

$$F(P) = F_W(P),$$

as was to be proved ²).

---

¹) See DE LA VALLÉE POUSSIN l.c. page 22.

²) A similar proof can be given in some special cases where Σ_i still contains only regular points but is not the sum of a number of closed sets.

3. In all the cases mentioned before it is supposed that the open set Ω has irregular boundary points, so that the DIRICHLET problem has not always a solution. If Ω has only regular boundary points, F and F_W are both identical with the solution of the DIRICHLET problem and there is no question at all. But our proofs do not assure that then in all cases $u^P(e) = \mu^P(e)$ (we have not proved generally that the representation (1) is inique; this is a new problem).

4. If it be assumed that $F = F_W$, we have, as is easily seen,

$$U^*(P; R) = G(P; R),$$

where G means the generalized (in the sense of WIENER) GREEN's function for Ω. From the symmetry of G it then follows that

$$\int \frac{du^P(e_Q)}{RQ} = \int \frac{du^R(e_Q)}{PQ}.$$

$$(P \subset \Omega; R \subset \Omega).$$

Dordrecht, April 28, 1939.

Physiology. — *Studies on phosphorus metabolism in normal and rachitic chickens with a radioactive phosphorus isotope. III. The distribution of the injected phosphorus in the organs, especially in the different parts of the leg bones.* By M. J. L. DOLS and B. C. P. JANSEN (Laboratory of Physiological Chemistry, University, Amsterdam) and G. J. SIZOO and G. J. VAN DER MAAS (Natuurkundig Laboratorium, Vrije Universiteit, Amsterdam). (Communicated by Prof. G. VAN RIJNBERK.)

(Communicated at the meeting of May 20, 1939.)

In previous papers [1]) experiments have been published on the phosphorus metabolism in normal and rachitic rats, by which the absorption, excretion, deposition in bone as well as the formation of lipin phosphorus were investigated with a radioactive phosphorus isotope as an indicator.

In this paper, we wish to discuss two series of experiments on the distribution of phosphorus in the leg bones and some organs of normal and rachitic chickens after intraperitoneal injection of labelled sodium phosphate. In both series, a fixed quantity of the active phosphate was injected as an aqueous solution of pH 7.2. After it had been established in rat experiments, that 20 hours after the injection the largest difference in the distribution exists between normal and rachitic rats, in these chicken experiments the animals were decapitated 22 hours after the injection and quickly sectioned. The leg bones then were dissected and cleaned. One of the leg bones of each bird was divided into three parts on the basis of the radiographs, namely the proximal epiphysis, the distal epiphysis and the diaphysis. In the first series with 8 chickens the two epiphysial parts of the bone were carbonized together in an oven at 200 degrees centigrade; the same was done with the diaphysis. Experimentally it was found, that no phosphorus was lost in this procedure. Two preparations now were made for the determination of the radioactivity in the manner described previously, namely a preparation of the two epiphysial parts together and a preparation of the diaphysial part of the leg bone. In the second series with 26 chickens, the leg bones were dried at 105 degrees centigrade for 24 hours, extracted with absolute alcohol for 48 hours and then dried again. Now, one of the leg bones of each bird was divided into three parts, namely the proximal epiphysis, the distal epiphysis and the diaphysis, which were carbonized separately. In this series four preparations of a leg bone of each chicken were made for the determination of the radioactivity, namely a preparation of the proximal epiphysis, of the distal epiphysis, of the diaphysis and of the alcohol extract of the whole leg bone.

---

[1]) Proc. Kon. Akad. v. Wetensch., Amsterdam, **40**, 547 (1937); Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 997 (1938); Nature, **139**, 1068 (1937); **141**, 77 (1938).

In our first experiment we had found, that there was a large difference in the phosphorylation of the injected phosphorus in the bones of normal and rachitic animals. Now, one could ask, is it only characteristic for the bones or also for the other organs, containing phosphorylating enzymes. Therefore, in this series it was also investigated, whether there is any difference between the phosphorus content and the distribution of injected phosphorus in the liver and the spleen of normal and rachitic chickens. So, preparations for the estimation of the radioactivity in these organs were made. In addition, a part of the residue of carbonization of the various parts of the leg bones and of the livers was used for the determination of the total phosphorus content according to FISKE and SUBBAROW.

The determination of the radioactivity in the different preparations was also done as described previously ²) with an ionization chamber, whereby the ionization current produced by the β-particles, emitted by the radioactive phosphorus in the bone preparations, was compared with the ionization current of a constant source, in this case the γ-activity of a piece of pitch blend.

The provisional figures of the first series with 5 normal and 3 rachitic chickens, which are tabulated in table I, show, that the phosphorus content of the carbonized residue in the epiphysial parts and in the diaphysial part is larger in the normal than in the rachitic chickens. Furthermore it was established, that both in the normal and in the rachitic chickens, the phosphorus content from the diaphysial part of the bone seems to be larger than that from the epiphysial parts of the same bone. With regard to the distribution of the active phosphorus administered it was observed, that both in the normal and in the rachitic chickens decapitated 22 hours after the injection of the active phosphorus, the quantity of the active phosphorus present in 1 mgm bone phosphorus was larger in the epiphysis than in the diaphysis. Furthermore, it was found, that both the epiphysis and the

TABLE 1.

	Number of chickens	Percentage phosphorus carbonized residue	Percentage of the active phosphorus administered present in 1 mgm of the bone phosphorus
Normal chickens	5		
Epiphysis		7.84	0.0255 ± 0.009
Diaphysis		11.10	0.0085 ± 0.003
Rachitic chickens	3		
Epiphysis		5.55	0.0619 ± 0.014
Diaphysis		8.61	0.0251 ± 0.008

---

²) G. J. SIZOO and C. P. KOENE, Physica **3**, 1053 (1936).

diaphysis of the rachitic birds contained a much larger quantity of the active phosphorus in 1 mgm bone phosphorus than the normal chickens do.

These experiments were continued and repeated with 26 birds in the second series mentioned before; the results are fully in accordance with those of the first series and produced in table 2.

As is shown in this table, the phosphorus content of the residue of carbonization, both in the epiphysial parts and in the diaphysial part from the leg bones is larger in the normal than in the rachitic chickens, whereas in both groups the diaphysial part of the leg bones seems to contain a larger phosphorus content than the epiphysial parts.

TABLE 2.

Percentage of phosphorus in the residue of carbonization from the leg bones.

	Normal Chickens			Rachitic Chickens		
	Proximal Epiphysis	Diaphysis	Distal Epiphysis	Proximal Epiphysis	Diaphysis	Distal Epiphysis
	8.62	12.81	9.14	4.78	10.10	5.71
	6.41	11.64	7.17	4.82	9.78	4.96
	8.30	13.13	8.26	5.34	10.45	5.84
	7.00	12.51	8.44	5.83	10.27	6.41
	9.33	12.94	7.58	6.23	10.16	6.18
	9.75	13.25	10.13	5.93	9.53	6.19
	10.39	14.55	10.34	4.17	9.05	4.66
	8.39	12.50	9.34	5.14	9.46	4.28
	9.40	13.12	9.53	5.23	9.78	5.51
	9.70	13.74	9.27	5.54	9.96	6.03
	8.79	13.99	9.44	5.19	9.73	5.13
	9.11	13.37	9.92			
	7.98	13.42	8.98			
	8.73	13.71	7.02			
Total	121.90	184.68	124.56	58.20	108.27	60.90
Mean	8.71	13.19	8.90	5.29	9.84	5.54
Standard Deviation	0.28	0.20	0.26	0.18	0.18	0.20

No difference exists in both groups between the phosphorus content of the proximal and the distal epiphysis.

A statistical treatment of the obtained dates established that the mentioned differences are clearly significant.

The difference between the phosphorus content of the diaphysis and the epiphysis in the group of the normal chickens is:
$D = 13.19 \pm 0.20 - 8.90 \pm 0.26 = 4.29 \pm 0.33$. The t-value according to FISHER 3) is 13, whereas the corresponding value for $P = 0.00$.

The same fact was established in the rachitic group; the difference between the diaphysis and epiphysis here was:

---

3) R. A. FISHER, Statistical Methods for Research Workers, 4th Ed. 1932.

$D = 9.84 \pm 0.18 - 5.54 \pm 0.20 = 4.30 \pm 0.27$; the t-value is 16 and the corresponding value for $P = 0.00$.

The difference between the epiphysial parts of the leg bones from the normal and the rachitic chickens is also significant; the same thing can be said from the diaphysis of these two groups.

The differences with their t-values and corresponding values for P according to FISHER are:

Proximal epiphysis:

$$8.71 \pm 0.28 - 5.29 \pm 0.18 = 3.42 \pm 0.33; \quad t = 10.4; \quad P = 0.00$$

Distal epiphysis:

$$8.90 \pm 0.26 - 5.54 \pm 0.20 = 3.36 \pm 0.33; \quad t = 10.2; \quad P = 0.00$$

Diaphysis:

$$13.19 \pm 0.20 - 9.84 \pm 0.18 = 3.35 \pm 0.26; \quad t = 13.0; \quad P = 0.00$$

With regard to the distribution of the active phosphorus in the bone, tabulated in table 3, it was found that both in the normal and in the rachitic group the two epiphysial parts of the leg bone contain more of the active phosphorus than the diaphysial part does. Furthermore, it was found, that the proximal epiphysis in both groups contains more of the active phosphorus than the distal epiphysis.

TABLE 3.

Percentage of the active phosphorus administered present in 1 mgm of the bone phosphorus.

	Normal chickens			Rachitic chickens		
	Proximal Epiphysis	Diaphysis	Distal Epiphysis	Proximal Epiphysis	Diaphysis	Distal Epiphysis
	0.0435	0.0169	0.0308	0.0622	0.0267	0.0442
	0.0808	0.0207	0.0568	0.0665	0.0196	0.0454
	0.0316	0.0081	0.0196	0.0593	0.0187	0.0448
	0.0748	0.0233	0.0630	0.0539	0.0183	0.0444
	0.0525	0.0154	0.0608	0.0697	0.0240	0.0315
	0.0332	0.0071	0.0233	0.0525	0.0203	0.0291
	0.0437	0.0153	0.0382	0.0724	0.0226	0.0477
	0.0456	0.0132	0.0327	0.0503	0.0182	0.0456
	0.0384	0.0068	0.0202	0.0679	0.0210	0.0393
	0.0307	0.0107	0.0206	0.0653	0.0218	0.0395
	0.0434	0.0090	0.0234			
	0.0287	0.0092	0.0233			
	0.0631	0.0187	0.0484			
	0.0315	0.0092	0.0246			
Total	0.6415	0.1836	0.4857	0.6200	0.2112	0.4115
Mean	0.0458	0.0131	0.0347	0.0620	0.0211	0.0411
Standard Deviation	0.0044	0.0014	0.0043	0.0024	0.0009	0.0021

The comparison of the figures of the normal and the rachitic chickens shows, that the different parts from the leg bones of the rachitic chickens contain more of the active phosphorus administered than the same parts of the normal group. With the exception of the difference between the distal epiphysis of the normal and the rachitic chickens all the other differences are clearly significant.

The statistical treatment of these figures shows the following results:

Normal chickens.

Difference between the proximal and distal epiphysis:

$$D = 0.0458 \pm 0.0044 - 0.0347 \pm 0.0043 =$$
$$= 0.0111 \pm 0.0062;\ t = 1.79;\ P = 0.08$$

Difference between the proximal epiphysis and the diaphysis:

$$D = 0.0458 \pm 0.0044 - 0.0131 \pm 0.0014 =$$
$$= 0.0327 \pm 0.0046;\ t = 7.1;\ P = 0.00$$

Difference between the distal epiphysis and the diaphysis:

$$D = 0.0347 \pm 0.0043 - 0.0131 \pm 0.014 =$$
$$= 0.0216 \pm 0.0045;\ t = 4.8;\ P = 0.00$$

Rachitic chickens.

Difference between the proximal and distal epiphysis:

$$D = 0.0620 \pm 0.0024 - 0.0411 \pm 0.0021 =$$
$$= 0.0209 \pm 0.0032;\ t = 6.6;\ P = 0.00$$

Difference between the proximal epiphysis and diaphysis:

$$D = 0.0620 \pm 0.0024 - 0.0211 \pm 0.0009 =$$
$$= 0.0409 \pm 0.0026;\ t = 15;\ P = 0.00$$

Difference between the distal epiphysis and diaphysis:

$$D = 0.0411 \pm 0.0021 - 0.0211 \pm 0.0009 =$$
$$= 0.0200 \pm 0.0023;\ t = 8.7;\ P = 0.00$$

Normal and rachitic chickens.

The difference between the values of the normal and rachitic chickens in favour of the rachitic animals are:

Difference between the proximal epiphysis of rachitic and normal animals:

$$D = 0.0620 \pm 0.0024 - 0.0458 \pm 0.0044 =$$
$$= 0.0162 \pm 0.005;\ t = 3.23;\ P = 0.00$$

Difference between the diaphysis of rachitic and normal animals:

$$D = 0.0211 \pm 0.0009 - 0.0131 \pm 0.0014 =$$
$$= 0.008 \pm 0.0017;\ t = 4.7;\ P = 0.00$$

Difference between the distal epiphysis of rachitic and normal animals:

$$D = 0.0411 \pm 0.0021 - 0.0347 \pm 0.0043 =$$
$$= 0.0064 \pm 0.0048;\ t = 1.33;\ P = 0.20$$

In the last case the difference cannot be regarded as significant.

The values for the quantity of phosphorus administered present in the alcohol extract of the leg bones are calculated on 100 mgm dried matter and not on 1 mgm bone-extract phosphorus, because the quantity of the bone extract available for the estimation of phosphorus according to Fiske and Subbarow was too small. Though the activity in this bone extract was low, it seems that the extract of the rachitic bones contains more of the active phosphorus than the extract of the normal chickens does.

However, the divergency in the individual results was very large, as was shown in the standard deviations of the means.

The mean percentage of the active phosphorus administered present in 100 mgm of the dried matter of alcohol extract of the whole legbone was for the normal chickens: 0.00159 ± 0.00053 %, whereas for the rachitic chickens this value was 0.00619 ± 0.00246 %.

The difference and corresponding values for t and P were:

$$D = 0.00460 \pm 0.00252;\ t = 1.83;\ P = 0.08$$

The statistical treatment of the figures shows that the significance of this difference is not quite convincing; nevertheless the probability is high that the difference in favour of the rachitic birds has not been caused by chance.

Besides the leg bones, in the second series the radioactivity was also estimated in preparations of the spleen and the livers. The spleen preparations contain hardly any activity; therefore, no determinations of the phosphorus content in the preparations have been done. However, the liver preparations were very active. The mean and the standard deviation of the phosphorus content of the carbonized residue as well as the mean and the standard deviation of the percentage of the phosphorus administered present in 1 mgm of the liver phosphorus are reproduced in table 4.

TABLE 4.

Percentage of the phosphorus in the carbonized residue of the liver		Percentage of the administered phosphorus present in 1 mgm of the liver phosphorus	
Normal chickens 1.93 ± 0.046	Rachitic chickens 1.79 ± 0.089	Normal chickens 0.0551 ± 0.0057	Rachitic chickens 0.0580 ± 0.0065

As is clearly shown by the figures, no significant difference exists between the phosphorus content of the carbonized residue from the normal and the rachitic chickens. The difference $D = 0.14 \pm 0.10$; the t value is 1.4 and the corresponding value $P = 0.18$.

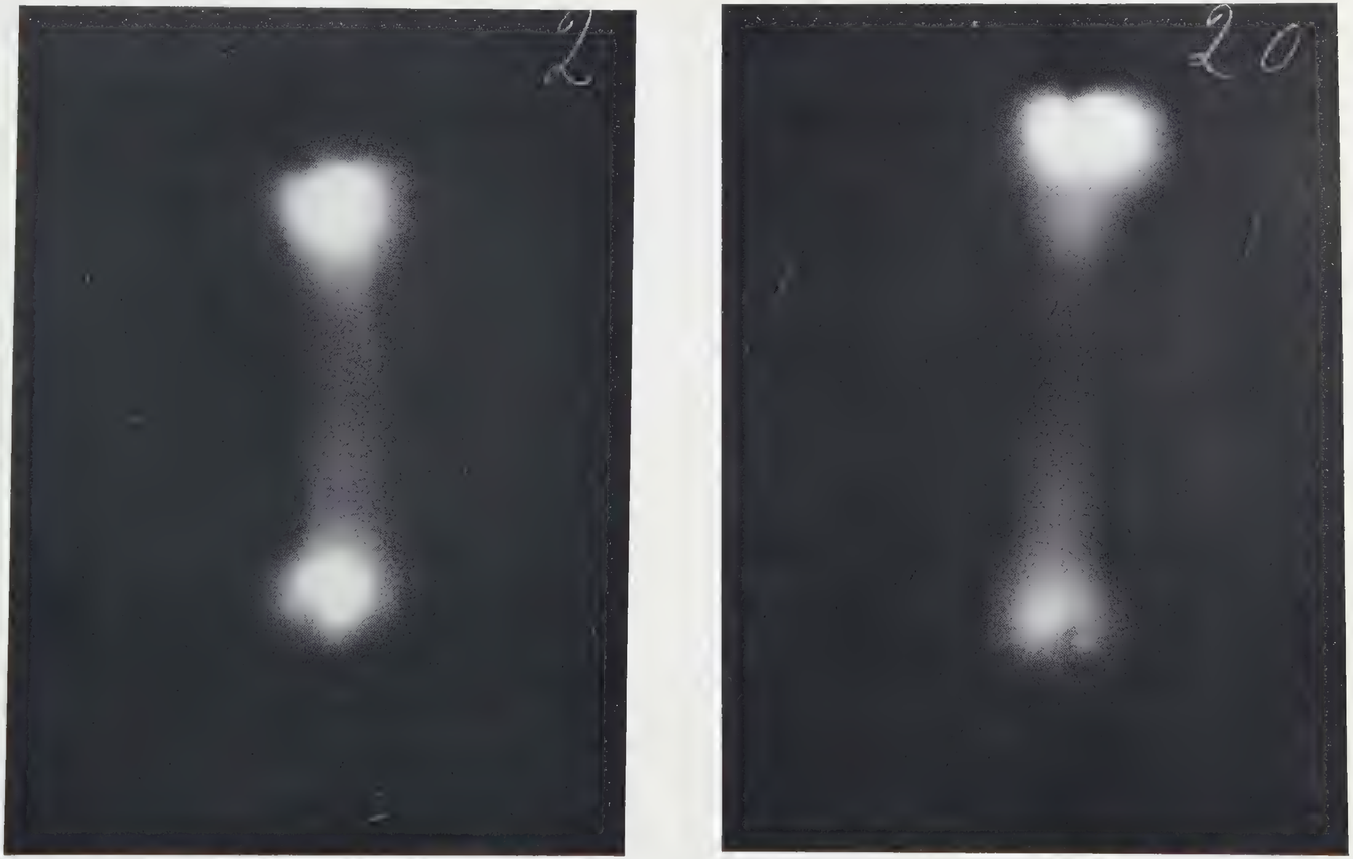
The same can be said of the percentage of the active phosphorus administered present in 1 mgm of the liver phosphorus; the difference is not significant.

The figures for the difference, standard deviation and corresponding t and P values are:

$$D = 0.003 \pm 0.0086;\ t = 0.35;\ P = 0.76.$$

So it has been established, that the bone seems to play a unique rôle in the phosphorus metabolism as has also been seen in the rat experiments; only in the bones a difference was found in the phosphorus metabolism; in the liver and the spleen no difference could be observed. In further

M. J. L. DOLS, B. C. P. JANSEN, G. J. SIZOO AND G. J. VAN DER MAAS: STUDIES ON PHOSPHORUS METABOLISM IN NORMAL AND RACHITIC CHICKENS WITH A RADIOACTIVE PHOSPHORUS ISOTOPE. III. THE DISTRIBUTION OF THE INJECTED PHOSPHORUS IN THE ORGANS, ESPECIALLY IN THE DIFFERENT PARTS OF THE LEG BONES.



"Auto-radiograph" of the tibia of a normal (20) and of a rachitic chicken (2).

experiments we wish to investigate also the muscle, the kidneys and the brain.

Summarizing the mentioned results, it may be concluded, that the figures convincingly show, that both in the normal and in the rachitic chickens the phosphorus metabolism is more intensive in the epiphysis than in the diaphysis of the leg bones.

With regard to the difference between the normal and the rachitic chickens the figures show, that generally the phosphorus metabolism is more intensive in the bones of the rachitic birds than in the bones of the normal chickens. These conclusions are fully in accordance with those obtained in our experiments on the formation of lipin phosphorus in normal and rachitic rats and also with the fact that the blood plasma and the bones of rachitic animals contain a larger amount of phosphatase than those of the normal animals [4]). In further experiments we now have started to investigate in vitro the influence of vitamin D on bone phosphatase.

The second leg bone of each chicken was not carbonized, but after being cleaned it was placed in both series on a double coated x-ray film. It remained on this film for some days according to the quantity of active phosphorus injected. The film then was developed; a clear picture "Autoradiograph" of the whole bone was visible as is shown by the accompanying photographs. It is convincingly shown in these, that the epiphysis contains a larger quantity of the active phosphorus than the diaphysis does. Hence the results of the measurements of the radioactivity of the different parts of the bones are confirmed by the obtained photographs.

Summary.

Experiments have been discussed in which particularly the distribution of phosphorus in the leg bones of chickens was investigated, with a radioactive phosphorus isotope as an indicator.

The figures obtained convincingly showed, that both in the normal and in the rachitic chickens the phosphorus metabolism is more intensive in the epiphysis than in the diaphysis of the leg bones, whereas the phosphorus metabolism in the bones of the rachitic birds is more intensive than in the bones of the normal chickens.

Moreover a method has been published by which it is possible to photograph the distribution of the phosphorus in the leg bones of animals.

We are much indebted to Prof. LAWRENCE from the University of California for a generous gift of radioactive phosphorus; to Mr. J. C. DE BACK, Mr. W. G. LINDENBERG and Mr. W. SOMBROEK for their assistance. Finally, we wish to express our heartfelt thanks to the "Vereeniging tot het Bevorderen van de Beoefening der Wetenschap onder de Katholieken in Nederland" and to the "Stichting voor Biophysica" for a grant for our researches.

---

[4]) H. D. KAY, Physiol. Reviews **12**, 384 (1932); J. Biol. Chem. **89**, 249 (1930).

Histology. — *Ueber die Beziehungen des Energiestoffwechsels zu Wachstum und Differenzierung* *). II. *Die Wirkung von Sauerstoff auf Wachstum und Entwicklung von Hühnerembryonen.* Von FRITZ DEMUTH. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of May 20, 1939.)

In der Mitteilung I. [1]) wurde darauf hingewiesen, dasz bei den ersten Teilungen des Eies die Atmung gesteigert ist (WARBURG [2])). Zu dieser Zeit findet keine Massenzunahme, kein reines Volumwachstum statt, sondern nur eine Verteilung der gleichen Zellmasse von einer Zelle auf mehrere, entsprechend kleinere Zellen, d.h. eine reine Differenzierung. Differenzierung geht also mit einem Mehrverbrauch von Sauerstoff einher. Erhöhtes Sauerstoffangebot hemmt die Glykolyse. Ohne Glykolyse gibt es kein Wachstum. Es wurde erwartet, dasz eine erhöhte Sauerstoffzufuhr das Wachstum von Hühnerembryonen hemmt, ohne die Differenzierung zu stören (obgleich POTT [3]) keine Wirkung von Sauerstoff auf die Entwicklung von Hühnereiern gesehen haben will). Umgekehrt müsste Hemmung der Atmung die Differenzierung hemmen, aber das Volumwachstum fördern.

Bei einigen Stichproben an Hühnereiern erwies es sich als schwierig, den Einflusz atmungshemmender Faktoren (HCN, Stickstoffatmosphaere) auf die Entwicklung des Embryo zu untersuchen, ohne ihn abzutöten oder von vornherein schwere Miszbildungen zu erzeugen. Die Wirkung des Sauerstoffmangels ist in der landwirtschaftlichen Praxis bekannt. U.a. fand AGGAZZOTTI [4]), dasz sich in einer Höhe von 3000 m ü.M. etwa ein Drittel der Hühnerembryonen gar nicht entwickelt und nur ganz wenige die zweite Woche erleben. Die Methode der Lackierung der Eier ist noch weniger geeignet, da bei ihr neben der Beschränkung der Sauerstoffaufnahme gleichzeitig die Kohlensäureabgabe verhindert ist.

Ich beschränkte mich deshalb darauf, die Aufzucht von Hühnereiern in einer erhöhten Sauerstoffatmosphäre zu versuchen. Eier von weiszen Leghornhühnern aus eigener, vom gleichen Muttertier stammenden Zucht wurden in Gläser von 1 Liter Inhalt gelegt, die durch paraffinierte Stopfen gut verschlossen waren. Durch Glasröhrchen wurde käuflicher Sauerstoff oder eine Mischung von 95 % Sauerstoff mit 5 % Kohlensäure, bei den Kontrollen Luft durchgeleitet. Die Gase passierten eine Vorlage mit Wasser. In den ersten Minuten wurden etwa 5 Liter Gas durch die Flaschen geschickt, in den folgenden Tagen etwa 1 Liter pro Stunde. Die Gläser und die Wasservorlagen standen im Thermostaten bei 38° C. Alle zwei Tage wurden die Eier gewendet. In bestimmten Abständen wurden Eier geöffnet, das Funktionieren des Herzschlages als Lebenszeichen kontrolliert, der Embryo von den Gefäszen abgeschnitten und schnell von den anhaftenden Häuten gereinigt. Embryonen mit Gewichten bis etwa zu 50 mgr wurden auf einer Torsionswage gewogen, schwerere in vorgewogene Gläschen mit 10 proz. Formalin geworfen und gewogen. Alle Embryonen wurden in der gleichen Weise fixiert, in Paraffin eingebettet, in Serien von 10 μ Dicke geschnitten und mit Hämalaun-Eosin gefärbt.

---

*) Dat ik dit onderzoek heb kunnen voltooien, dank ik aan de hulp mij bewezen door het Academisch Steunfonds te Amsterdam, in het bijzonder Prof. FRIJDA. Voor het verschijnen ben ik Prof. BOEKE zeer verplicht.

[1]) DEMUTH, Roux' Arch., **130**, 340 (1933).

[2]) WARBURG, Z. f. physiol. Chem., **66**, 305 (1910).

[3]) POTT, Arch. f. d. ges. Physiol., **31**, 268 (1883).

[4]) AGGAZZOTTI, Roux' Arch., **40**, 65 (1914).

Der Sauerstoffpartialdruck in den Gläsern ist nicht genau festgelegt, aber praktisch liegt nach wenigen Minuten eine fast reine Sauerstoffatmosphäre vor. Der Sauerstoffgehalt der Luftkammer des Eies wurde nicht bestimmt. Nach AGGAZZOTTI ¹) ist er bei befruchteten Eiern in Luft während der ersten Bebrütungstage dem Sauerstoffgehalt der umgebenden Luft gleich, sinkt aber vom 6. Tag ab bis auf 13,15 % ab, so dasz man schon im Normalfall von einem relativen Sauerstoffmangel im bebrüteten Ei sprechen kann. In 3000 m Höhe wird die Luftkammer kleiner, der Sauerstoffgehalt gleicht sich der umgebenden Luft an. Umgekehrt dürfte also bei erhöhtem $O_2 = PD$ der Umgebung auch derjenige im Ei erhöht sein. Vergleiche mit den widersprechenden Angaben über Ab- oder Unabhängigkeit des Sauerstoffverbrauches vom $O_2 = PD$ scheinen mir untunlich. Sie sind sicher von Material und Methode beeinfluszt. Es genügt mir zu zeigen, welche eindeutigen Veränderungen eintreten bei der Entwicklung des Hühnereies, wenn nichts geändert wird, als der Sauerstoffgehalt der Umgebung.

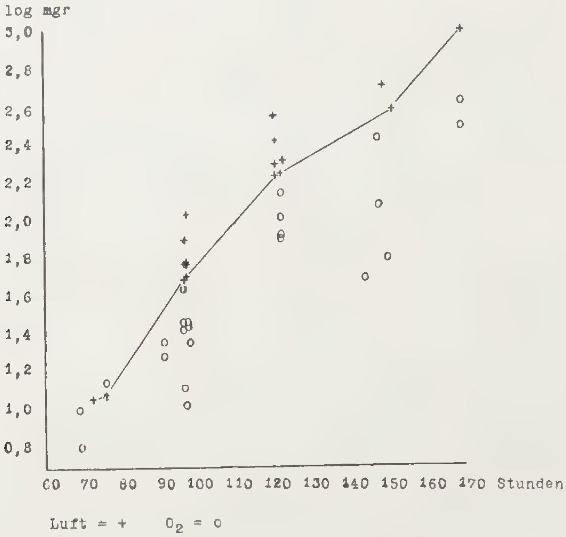
Gewicht.

Abgesehen von den ersten 3 Tagen ist das Gewicht der Sauerstofftiere immer kleiner als das der Kontrollen. Es kann bis auf ein Achtel gegenüber den Kontrollen heruntergehen. Da das Wachstum in logarithmischer Progression verläuft, wurden in der Tabelle die Logarithmen der Gewichte angegeben und in einer Kurve eingezeichnet. Aus ihr ergibt sich die Streuung der normalen Gewichte und es läszt sich ablesen, um wie viele Stunden die Versuchstiere in ihrem Gewicht zurückgeblieben sind gegenüber den Kontrollen und zwar den Kontrollen mit dem *niedrigsten* Gewicht des betreffenden Alters, was als minimale Differenz bezeichnet werden

---

¹) AGGAZZOTTI, Roux' Arch., **40**, 65 (1914).

mag. Aus der minimalen Differenz wurde das Alter der Sauerstofftiere nach ihrem Gewicht abgelesen und dem wirklichen Alter in der Tabelle



gegenübergestellt. Die angeführten Zahlen stellen also ein *Höchst*alter der Tiere, das ihrem Gewicht entsprechen würde, dar.

Aeuszere Gestalt.

Der Entwicklungsstand der Embryonen läszt sich aus der äuszeren Gestalt nicht gut erkennen. Ein Vergleich mit den Normentafeln oder zwischen Experiment- und Kontrolltier ist bei den erheblichen Gröszenunterschieden kaum möglich, die auf Abb. 1 sehr deutlich zu sehen sind. Man kann nur ganz allgemein sagen, dasz manche Sauerstofftiere, trotzdem sie wesentlich kleiner sind als die Kontrollen, keinen deutlichen Unterschied im Entwicklungsgrad erkennen lassen, manche aber makroskopisch sicher zurückgeblieben zu sein scheinen (z.B. vergleiche man die hinteren Extremitäten von 50 und 51). Aber die makroskopische Betrachtung kann sehr täuschen, wie die mikroskopische Untersuchung gezeigt hat (z.B. beim Auge von 50 und 51).

Mikroskopische Befunde.

Bei den mikroskopischen Untersuchungen konnte weniger das erste Auftreten neuer Organe berücksichtigt werden. Dies ist oft schwierig und schon normalerweise ist die Zeitspanne für die Möglichkeit des Auffindens neugebildeter Organe ziemlich grosz. Daher wurde mehr auf den Entwicklungszustand der einzelnen deutlich erkennbaren Organe geachtet, besonders solcher mit relativ einfachem, in verschiedenen Schnittebenen gleich erscheinendem Bau (z.B. die Linse des Auges), und auch auf leicht übersehbare Beziehungen zwischen Organbestandteilen, die von verschie-

FRITZ DEMUTH: UEBER DIE BEZIEHUNGEN DES ENERGIESTOFFWECHSELS ZU WACHSTUM UND DIFFERENZIERUNG. II. DIE WIRKUNG VON SAUERSTOFF AUF WACHSTUM UND ENTWICKLUNG VON HÜHNEREMBRYONEN.

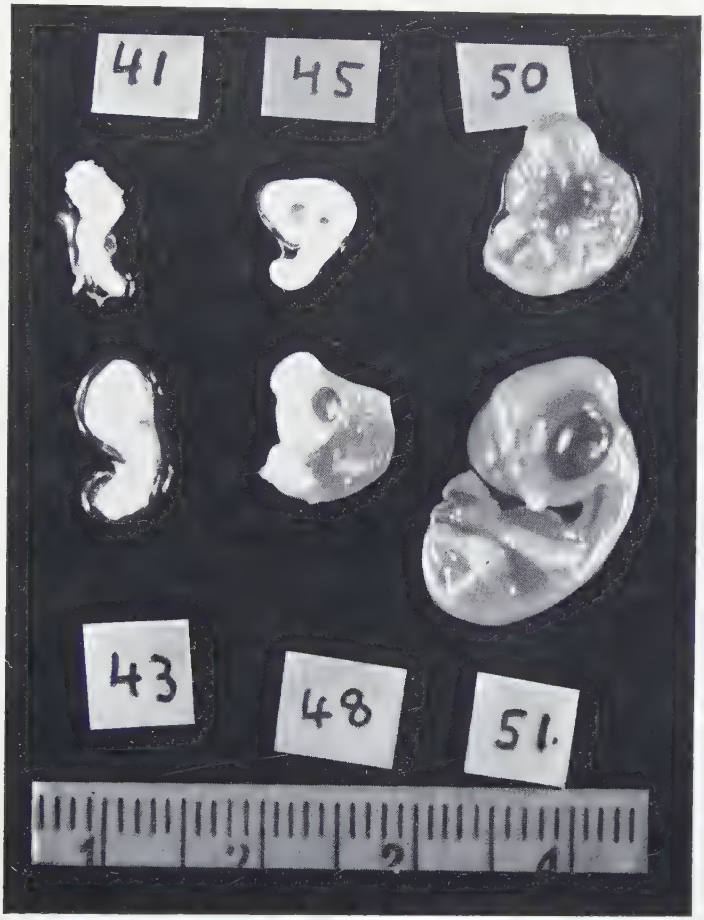


Abb. 1.

denen Keimblättern abstammen. Vor allem wurden die Drüsen mit innerer Sekretion beachtet. Es wurde erwartet, Verschiebungen im Verhältnis des Entwicklungszustandes verschiedener Organe zu einander und bestimmter Teile innerhalb eines Organes zu einander zu finden.

Im allgemeinen sind alle Organe bei den Tieren mit niedrigerem Körpergewicht kleiner als bei den Kontrollen. Eine Ausnahme macht nur die Epiphyse. Die Zahl der die Organe aufbauenden Zellen ist geringer. Bei mehreren Zellschichten kann die Zahl dieser Schichten kleiner sein. Infolgedessen sehen die Organe der Sauerstofftiere oft zierlicher aus. Ueber die Zellgröszen wage ich nach den Erfahrungen, die ich mit der Messung von Zellen in lebendem (in vitro) und dann in fixiertem Zustand gemacht habe, kein Urteil abzugeben.

Augen.

Die Augen sind ein sehr brauchbares Vergleichsobjekt. Bei den Sauerstofftieren des 3. Tages ist das Linsenbläschen schon geschlossen (21, 22, 16) oder fast geschlossen (15), während es bei der Kontrolle (26) noch weit offen ist. Allerdings sind die Gewichte dieser Embryonen noch nicht sehr verschieden. Z.T. mag das an Wägefehlern liegen, die dadurch entstehen, dasz beim Praeparieren trotz grösztmöglicher Geschwindigkeit doch Wasserverluste eintreten. Immerhin ist auch bei demjenigen Tier, dessen Gewicht nur die Hälfte von dem der anderen beträgt (22), die Linsenblase geschlossen. Die Trennung vom Epidermisepithel ist überall dem Alter entsprechend deutlich. Nur bei 22 besteht an einer Stelle eben noch eine lockere Verbindung.

Am 4. Tage hat bei 3 Sauerstofftieren (23, 40, 31) das hintere Linsenepithel das vordere bereits erreicht, was bei keiner Kontrolle der Fall ist. Die erste Mesenchymeinwanderung zwischen Linse und Epidermisepithel ist bei den Sauerstoffembryonen 1 Mal (40) eben im Beginn und 1 Mal (31) deutlich festzustellen, während sie bei den Kontrollen immer noch fehlt. Die Sauerstofftiere, bei denen die Augenentwicklung gegenüber den Kontrollen am weitesten voraus ist, müssten nach ihrem Gewicht 7 bis 12 Stunden jünger sein als das kleinste Kontrolltier, aber sogar 17 bis 21 Stunden jünger als das gröszte.

Am 5. Tage sind mit Ausnahme von 45, bei dem sich noch eine kleine Lücke in der Linse findet, keine Unterschiede in der jetzt deutlich werdenden Irisbildung und der Bildung des Sehnerven zu erkennen. Ueberall ist die jetzt entstehende Nervenfaserschicht der Retina deutlich. Dabei ist dem Gewicht nach 45 um mindestens 14, 5 um mindestens 15 Stunden jünger als die Kontrollen.

Unter den Sauerstofftieren des 6. Tages nimmt 25 eine Sonderstellung ein. Es ist seinem Gewicht nach um volle 2 Tage jünger, als seinem wirklichen Alter entspricht. Das Auge ist sehr primitiv. Die Linse entspricht etwa dem Alter aus dem Gewicht. Der Sehventrikel ist ungewöhnlich weit offen. Andrerseits ist die Iris weit entwickelt. Doch findet sich an ihrem retinalen Teil eine blasige Miszbildung, während in der Nähe des Sehnerveneintrittes die Retina eine mehrkammrige Ausstülpung bildet. Die Augenentwicklung ist also nicht nur gehemmt, sondern im Sinne einer Miszbildung gestört. — Die Augen von 6 und 52 gleichen der Kontrolle 29.

Ebenso sind die Augen der Tiere des 7. Tages alle gleich weit entwickelt, nur sind die kleineren Augen der Sauerstofftiere entsprechend zierlicher. Dabei ist 7 um einen ganzen Tag dem Gewicht nach zurück. Man würde bei makroskopischer Betrachtung allein nicht glauben, dasz die Augen von 50 und 51 sich nur durch die Grösze unterscheiden.

Wenn 25 auszer Betracht gelassen wird, so sind also die Augen bei den Sauerstoffembryonen stets weiter entwickelt, als dem Gewicht der Tiere entsprechen würde, ja z.T. sind sie auch absolut weiter entwickelt als die gleichaltrigen Kontrollen, selbst wenn sich die Körpergewichte wie 1 : 3 verhalten.

Hypophyse.

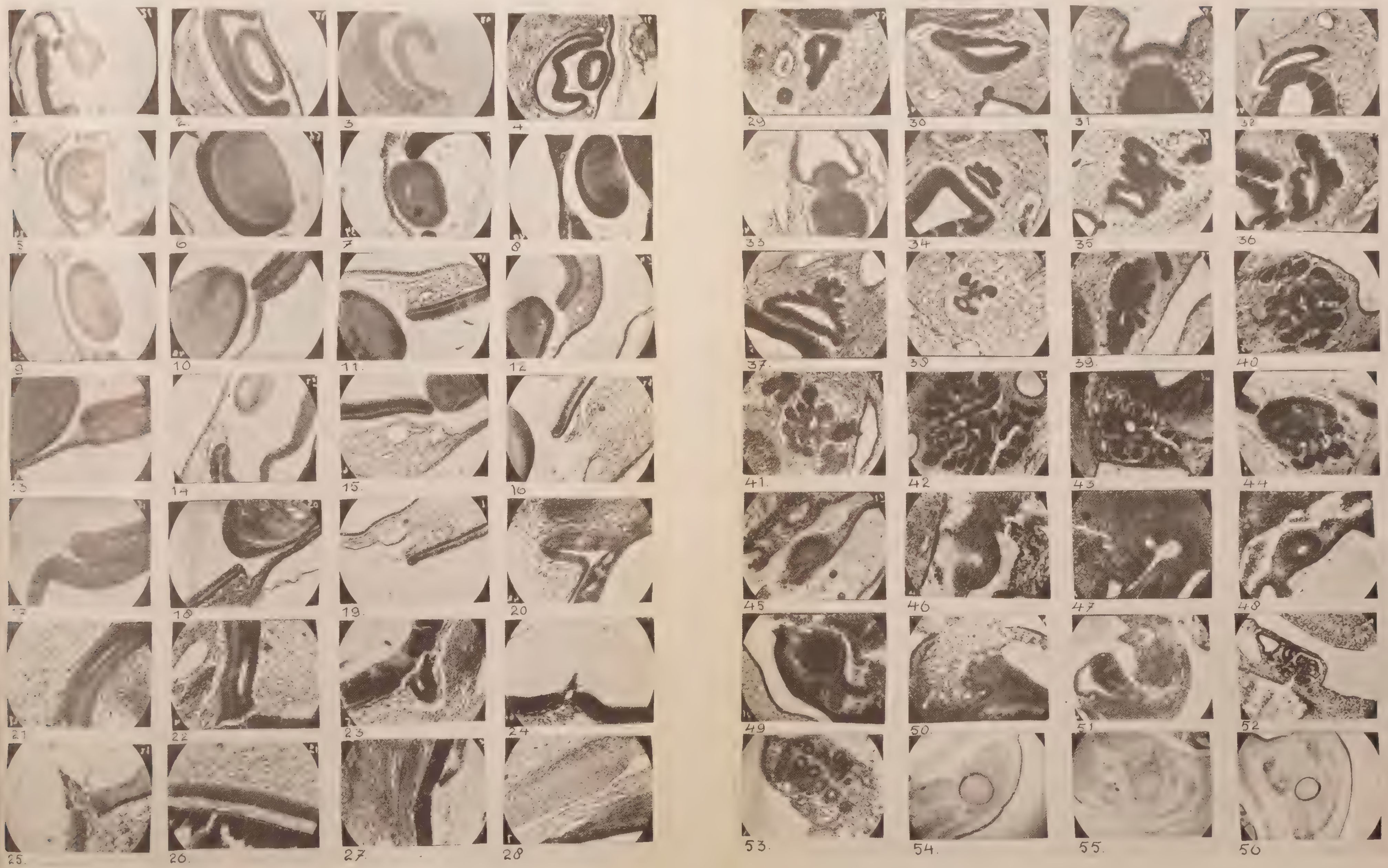
Die Hypophyse ist bei allen Sauerstofftieren an allen Tagen eben so weit entwickelt wie bei den gleichaltrigen Kontrollen. Manchmal ist sie sogar weiter fortgeschritten. So ist besonders der 4. Tag zu beachten (z.B. 31 gegenüber 34), an dem sich häufiger die ersten Drüsenbildungen in Gestalt von Knospen zeigen als bei den Kontrollen. In den nächsten Tagen ergeben die Schnitte durch das mehrkammerige Organ rosettenartige Bilder, deren Kompliziertheit ein Masz für den Fortschritt der Entwicklung ist. Besonders auffallend ist 25: Die Hypophyse ist sehr klein. Aber sie scheint nur im Ganzen sehr zierlich gebaut, im Prinzip aber eben so kompliziert zu sein wie die gleichaltrigen Kontrollen. Z.B. ist der Hypophysengang zu einem ganz schmalen zentralen Spalt geschlossen. Deutlich sind 5 verschiedene Lappen zu erkennen (also Vorraum, Ventrikelsäckchen, 2 Seitenlappen und Rathkesche Tasche). Der Vergleich der Abbildungen von 25 und 29, die natürlich im gleichen Maszstab aufgenommen sind, ist um so bestechender, wenn man bedenkt, dasz 25 fast acht Mal so klein ist wie 29. Auch der Gröszenunterschied der beiden Organe nach dem Schnittbild beträgt etwa das Achtfache.

Epiphyse.

Das Gehirn eignet sich nicht gut zur vergleichenden Untersuchung. Die Unmöglichkeit bei mehreren Embryonen eine einigermaszen gleiche Schnittrichtung des verhältnismäszig groszen Organes zu erhalten, erschwert die Beurteilung der Entwicklung der einzelnen Hirnteile. Die Dicke der Zellschichten oder der weiszen Substanz ist nicht sicher abschätzbar, die Auffindung des ersten Auftretens etwa von Pyramidenzellen sehr schwierig. Im ganzen verläuft auszerdem die Entwicklung des Gehirns bei den Hühnern sehr langsam und nicht sprunghaft.

Ein sehr gutes Beobachtungsobjekt ist aber die Epiphyse, besonders die Epiphyse im engeren Sinne (also die hintere). Sie ist zum ersten Male mit

FRITZ DEMUTH: Ueber die Beziehungen des Energiestoff-wechsels zu Wachstum und Differenzierung. II. Die Wirkung von S[illegible]f Wachstum und Entwicklung von Hühnerembryonen.



VERZEICHNIS DER FIGUREN VON ABB. 2.

[illegible]
1) Nr. 21. O_2 69 St.
2) Nr. 22. O_2 69 St
3) Nr. 26 L. 76 St
4) Nr. 16. O_2 76 St
5) Nr. 24. O_2 91 St.
6) Nr. 43. L. 96 St.
7) Nr. 34. L. 97 St.

Mesenchymale Einwanderung
8) Nr. 40. O_2 96 St.
9) Nr. 31. O_2 97 St.

Iris:
10) Nr. 48. L. 121 St.
11) Nr. 39. L. 121 St.
12) Nr. 45. O_2 121 St.
13) Nr. 47. O_2 121 St.
14) Nr. 25. O_2 144 St.
15) Nr. 6. O_2 147 St.
16) Nr. 29. L. 151 St.
17) Nr. 51. L. 168 St.
18) Nr. 50. O_2 168 St.
19) Nr. 7. O_2 168 St.

Augenstiel:
20) Nr. 48. L. 121 St.
21) Nr. 37. L. 121 St.
22) Nr. 47. O_2 121 St.
23) Nr. 5. O_2 121 St.

Pecten:
24) Nr. 6. O_2 147 St.
25) Nr. 29. L. 151 St.

Skleraknorpel:
26) Nr. 51. L. 168 St.
27) Nr. 50. O_2 168 St.
28) Nr. 7. O_2 168 St.

Hypophyse:
29) Nr. 23. O_2 91 St.
30) Nr. 43. L. 96 St.
31) Nr. 34. L. 97 St.
32) Nr. 40. O_2 96 St.
33) Nr. 41. O_2 96 St.
34) Nr. 31. O_2 97 St.
35) Nr. 39. L. 121 St.
36) Nr. 45. O_2 121 St.
37) Nr. 5. O_2 121 St.
38) Nr. 25. O_2 144 St.
39) Nr. 6. O_2 147 St.
40) Nr. 53. L. 148 St.
41) Nr. 29. L. 151 St.
42) Nr. 51. L. 168 St.
43) Nr. 50. O_2 168 St.
44) Nr. 7. O_2 168 St.

Schilddrüse:
45) Nr. 15. O_2 76 St.
46) Nr. 26. L. 76 St.
47) Nr. 34. L. 97 St.
48) Nr. 41. O_2 96 St.
49) Nr. 31. O_2 97 St.

Pankreas:
50) Nr. 44. L. 96 St.
51) Nr. 42. O_2 96 St.

Niere:
52) Nr. 43. L. 96 St.
53) Nr. 31. O_2 97 St.

Femur.
54) Nr. 51. L. 168 St.
55) Nr. 50. O_2 168 St.
56) Nr. 7. O_2 168 St.

Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLII, 1939

Sicherheit bei dem 3 Tage alten Sauerstofftier 16 zu finden. Die erste Andeutung einer Drüsenbildung ist am 4. Tage wieder bei einem Sauerstofftier feststellbar. Am 5. Tage ist die Drüsenentwicklung bei 2 von 4 Sauerstofftieren (45 und 47) weiter vorgeschritten als bei der am weitesten entwickelten Kontrolle (48). Sowohl am 5. wie am 6. Tage scheint das Organ je ein Mal (52 und 7) nicht nur gleich weit ausgebildet zu sein wie bei den Kontrollen, sondern auch im Verhältnis zur Grösze der Sauerstofftiere gröszer. Bei 25 ist es etwa auf dem Standpunkt des vorhergehenden Tages, also in der Mitte zwischen Alter und Gewicht. Es ist sicher kein Zufall, dasz die Epiphyse bei den Sauerstofftieren besonders weit entwickelt ist.

Vom 6. Tag ab ist die Bildung des Plexus Choreoideus gut zu sehen. Sie ist bei den Sauerstofftieren eben so weit entwickelt wie bei den Kontrollen.

Schilddrüse.

Die mediane Schilddrüsenanlage schlieszt sich am 3. Tage bei dem Sauerstofftier früher ab, als bei der Kontrolle. Im späteren Stadium ist kein Unterschied zwischen Sauerstoff- und Kontrolltieren zu finden, auch bei 25 nicht. Bei 7 ist sogar zum ersten Male eine Follikelbildung zu sehen, die bei den beiden anderen gleichaltrigen Tieren nicht zu erkennen ist.

Die Nebenschilddrüsen habe ich eben so wie die Nebennieren nicht oft genug gut beobachten können, um über sie etwas aussagen zu können.

Pankreas.

Beim Pankreas ist kein sicherer Unterschied zwischen Sauerstoff- und Kontrolltieren zu finden. Das Erkennen der Inseln ist so schwierig, dasz man bei Tieren, bei denen sie nicht gesehen worden sind, doch nicht ausschlieszen kann, dasz sie nicht doch vorhanden sind.

Leber.

Im Gegensatz zu den Drüsen mit innerer Sekretion sind Leber und Niere für unsere Zwecke weniger geeignete Objekte. Sie werden vom Organismus frühzeitig in voller Funktion gebraucht. Die zahlreichen Einzelteilchen, aus denen sie bestehen, sind daher schon früh, d.h. hier am 3. Beobachtungstage, voll entwickelt. Es ändert sich an der Bildung der Leberbälkchen, die schon einen typischen Bau zeigen, vom 3. Tage ab prinzipiell nichts mehr. Nur die Zahl der Balken und Läppchen und damit des ganzen Organes nimmt entsprechend dem Körperwachstum zu, was sich aber nicht abschätzen läszt. Auffallend ist, dasz bei den Sauerstofftieren oft gröszere und stärker gefüllte Bluträume finden als bei den Kontrollen, was übrigens auch für die Nieren gilt. Man sollte annehmen, dasz sich der Organismus gegen das Sauerstoffüberangebot durch Bildung von weniger Erytrocyten wehrt. Vielleicht ist die Anhäufung in Leber und Niere als eine Einsperrung von roten Blutkörperchen in einem Depot anzusehen. Gallenblase, Ductus choledochus und Ductus hepaticus werden früh angelegt und erscheinen bei Sauerstoff- und Kontrolltieren etwa zur gleichen Zeit.

Nieren.

Die ersten Glomeruli sind sehr früh zu finden und bei allen Tieren in gleicher Weise ausgebildet. Die Kompliziertheit ihres Baues, die Zahl der Schlingen oder der Kanälchen ist schwer abzuschätzen. Das Organ ist also für Vergleiche auch wenig brauchbar. Auffallend war nur der primitive Bau der Glomeruli von 25.

Femur.

Bei den Extremitäten ist die allgemeine Formgebung schwer zu beurteilen. Sie sind bei den Sauerstofftieren immer wesentlich kleiner und oft primitiver. Die Zeitspanne, innerhalb deren schon bei den Kontrollen in der zentralen Zellanhäufung z.B. des Femur mit Sicherheit Knorpel zu erkennen ist, ist sehr grosz. Man kann daher keinen all zu groszen Unterschied in der Knorpelentwicklung vom 4. bis 7. Tage zwischen Sauerstoff- und Kontrolltieren angeben. Nur 25 ist wieder viel primitiver. Die beginnende Verknöcherung ist dagegen sehr genau festzustellen. Sie ist am 7. Tage bei allen 3 Tieren zu sehen, aber gerade bei dem kleinsten Sauerstofftier (7) mit seinem sehr viel zierlicheren Femur besonders deutlich. Hier ist also die Knochenentwicklung, wenn man nach dem Gewicht urteilt, um mindestens einen Tag voraus. Auch würde man aus dem makroskopischen Vergleich nicht glauben, dasz die Knochenbildung bei den Sauerstofftieren mindestens eben so weit vorgeschritten ist, wie bei der Kontrolle.

Lunge.

Die Lunge bleibt lange unentwickelt. Erst vom 6. Tage ab wird der Brochialbaum stärker ausgebaut. Man hat hier den Vorteil, die Zahl der Bronchien ziemlich genau angeben zu können. Mit Ausnahme von 25 ist sie bei Sauerstoff- und Kontrolltieren bis zum 6. Tage gleich, am 7. Tag ist der Brochialbaum der Kontrolle stärker verästelt.

Andere Organe.

Die übrigen Organe sind für Vergleiche nicht geeignet, teils wegen der Art ihrer Bildung, wie beim Herzen (es läszt sich z.B. nicht entscheiden, ob eine Septumöffnung „schon" oder „noch" besteht), teils wegen zu langsamer Entwicklung, die keine scharfen Zeitpunkte bietet, wie bei Magen und Darm (z.B. das Auftreten bestimmter Drüsenzellen). Den Genitalapparat habe ich nicht untersucht, weil die Zahl der Tiere, die schon nicht sehr grosz ist, noch kleiner geworden wäre. Für die Betrachtung des Gehörapparates gelten die gleichen Schwierigkeiten bezüglich der Schnittführung wie beim Gehirn. Für das haematopoetische System und den Bewegungsapparat konnte ich keine geeigneten Anhaltspunkte finden. Die Thymusdrüse habe ich leider nicht genügend analysieren können.

Nr.	Luft oder O_2	Gewicht		Zeit St.	Zeit nach Gew.	Augen	Hypophyse vordere Anlage	Epiphyse	Schilddrüse	Pankrass	Leber	Urnieren	Femur	Lunge	
		mgr.	log mgr												
21	O_2	10.12	1.01	69	69	Linsenbläschen geschlossen, überall vom Epidermisepithel getrennt	offener, einschichtiger Schlauch	—				einfache Kanälchen			
22	O_2	6.30	0.80	69	69	Linsenbläschen geschlossen, an einer Stelle noch in lockerer Verbindung mit dem Epidermisepithel	do.	—				do.		beiderseits 1 Bronchialknospe	
26	L	12.00	1.08	76	76	Linsenbläschen teilweise noch sehr weit offen	do.	—	med. noch nicht geschlossene Rinne		wenige Balken	do.		do.	
15	O_2	12.30	1.09	76	76	1. Linsenbläschen geschlossen, 2. auf 3 Schnitten noch offen	do.	—	med. u. lat. fast abgeschlossene Schläuche		ziemlich kompliz. Balkensystem			do.	
16	O_2	14.00	1.15	76	76	Linsenbläschen geschlossen, von Epidermisepithel getrennt	do.	deutliche Ausstülpung	med. geschl., lat. offen		wenig Balken, groszer Blutraum			do.	
23	O_2	19.76	1.30	91	83	Hinteres Linsenepithel erreicht überall vorderes, weiter Sehventrikel, Mesenchymeinwanderung zwischen Linse und Epidermis	mehrschichtiger Schlauch, beginnende Knospung	—	m. u. l. geschlossen			Glomeruli +		je 1 Bronchus	
24	O_2	24.03	1.38	91	85	Hinteres Linsenepithel erreicht nicht überall vorderes, keine Mesenchymeinwanderung	mehrschichtiger Schlauch	—	do.			do.		do.	
43	L	79.0	1.90	96	106	Hinteres Linsenepithel erreicht nicht überall vorderes, keine Mesenchymeinwanderung	etwas knospig	einfache Ausstülpung	do. ziemlich grosz	deutliche Schläuche	Gallenblase +	do.		do.	
44	L	57.0	1.76	96	100	do.	glatter, mehrschichtiger Schlauch	do.	do.	do.	do.	do.		—	
28	L	50.4	1.70	97	97	—	—	—	do.	do.	do.			do.	
34	L	47.2	1.67	97	97	Hinteres Linsenepithel erreicht nicht überall vorderes, keine Mesenchymeinwanderung	noch ganz breit offene Rinne zum Rachen	längerer Schlauch	lat. noch etwas offen	do.	primitiver	do.		do	
40	O_2	29.70	1.47	96	89	Hinteres Linsenepithel erreicht überall vorderes, beginnende Mesenchymeinwanderung	etwas knospig	do.	abgegrenzt		Gallenblase +	do.		do.	
41	O_2	25.41	1.41	96	87	Hinteres Linsenepithel erreicht nicht überall vorderes, keine Mesenchymeinwanderung	noch breite Rinne, aber weiter entwickelt als 34	Schlauch mit kleiner Knospe	m. z.T. geschlossen, 1. gut abgegrenzt	eben erkennbar	do. grosze Bluträume	do.		do.	
42	O_2	42.76	1.63	96	95	do.	etwas knospig	wie 43	gut abgegrenzt	deutlich wie 44	G.B. + wohl ausgebildetes Balkensystem	do.		do.	
31	O_2	22.47	1.35	97	85	Hinteres Linsenepithel erreicht überall vorderes, deutliche Mesenchymeinwanderung	stark knospig	wie 43	do. grosz	do.	do.	do.		do.	
48	L	190.0	2.28	121	126	Mesenchymaler Iristeil verdickt, Sehventrikel des Augenstiels offen	Rosettenartige Schläuche	deutliche Drüsenbildung, Gang fast geschlossen	gleichförmige, kuglige Zellhaufen				zentrale Zellanhäufung, beginnende Knorpelbdg.	do.	
49	L	167.0	2 22	121	121	do.	do.	do.	do.				do.	do.	
37	L	249.0	2.40	121	137	do.	ziemlich stark entwickelte Rosette	einfacher, mehr knospig	do.	Insel +			Knorpel deutlich	do.	
38	L	167.0	2.22	121	121	do.	primitiver, knospiger Schlauch	wie 48, Gang weiter offen	do.	do.			kein Knorpel	do.	
39	L	197.6	2.30	121	128	do.	komplizierte Rosette	wie 48	do.	do.			zentrale Zellanhäufung	do.	
46	O_2	103.6	2.02	121	112	do.	Schlauch mit Knospen	wie 37	—	do.			Knorpel nicht sicher, aber deutliche Abgrenzung	do.	
45	O_2	85.4	1.93	121	107	Noch kleine Lücke in der Linse, sonst wie 37	komplizierte Rosette	mehr Drüsenschläuche als bei 48					do., primitiv	do.	
47	O_2	140.0	2.15	121	117	wie 49	komplizierte Schläuche	do.	do.	do.			Knorpel deutlich	do.	
5	O_2	78.6	1.90	121	106	wie 37	kleine Rosette und viele Knospen an weitem Schlauch	einfacher Schlauch	do.	do.			wie 39	do.	
53	L	492.0	2.69	148	156	Iris setzt sich von der Cornea ab, Pecten, Knorpel in der Sclera	stark verzweigtes Schlauchsystem, zentraler Spalt	komplizierte Drüse schmaler Stiel	palisadenartig um Gänge gestellte Zellen				Verdichtung der Knorpelrander	je 5—6 Bronchien	Plexus chorioid. eben angedeutet
54	L	360 0	2.56	148	151	kein Knorpel	wie 53, offener Gang	do.	—	do.			Knorpelrand verstärkt	je 4 Bronchien	nicht deutlich
29	L	356.4	2.55	151	151	Iris beginnt sich abzusetzen, Pecten, kein Knorpel	etwas weniger, sonst wie 53	do.	do.	do.			keine Verdichtung	je 5 Bronchien	do
25	O_2	47.2	1.67	144	96	sehr primitiv, in der Linse noch kleine Lücke, weit offener Sehventrikel, Nervenfaserschicht nicht sicher, mesenchymaler Iristeil weit vorspringend, blasige Miszbildung an der Iris und an der Retina beim Sehnerveneintritt	sehr zierliche, aber stark verzweigte Rosette, schmaler, zentraler Spalt	einfache Drüsenbildung wie 38	do., nur kleiner		sehr primitiv G. Bl. +	Glomeruli sehr einfach	zentrale Zellanhäufung, sehr primitiv	je 1 Bronchus	—
52	O_2	121.6	2.08	148	115	Iris abgesetzt, etwas plump, Pecten, kein Knorpel	wie 29	komplizierte Drüse, grosz, schmaler Stiel	—	do.			noch keine Verdichtung der Knorpelrander	je 4 Bronchien	do.
6	O_2	254.2	2.41	147	138	Iris setzt sich ab, Pecten, etwas kleiner, kein Knorpel	komplizierte Rosette, z.T. schmaler Spalt	wie 53, zierlich, sehr kompliziert	wie 53	do.			Verdichtung der Knorpelrander	je 4 Bronchien	deutlich
51	L	950.0	2.98	168	168	Iris deutlich abgesetzt, Cornea dick, Knorpel in der Sklera	sehr kompliziert	wie 53, nur gröszer	komplizierte Balkenbildung				beginnende Knochenbildung	stark verzweigter Bronchialbaum	do.
50	O_2	402.8	2.61	168	153	Iris und Cornea zierlicher, Knorpel	viel kleiner aber sehr kompliziert	wie 51	etwas weniger zellreich			viel kleiner weitmaschiger, blutreicher	do.	weniger verzweigt	do.
7	O_2	303.6	2.48	167	144	zierlicher aber genau so weit wie 51	kleiner, sonst fast wie 51	wie 51, grosz	Follikelbildung				Knochenbildung deutlich	[illegible]	[illegible]

Schluszfolgerungen.

Bei Aufzucht von Hühnerembryonen in einer Sauerstoff-reichen Atmosphäre sind die Tiere einer bestimmten Altersstufe immer im Gewicht gegenüber den gleichaltrigen Kontrollen zurückgeblieben. Demgegenüber bleibt die Differenzierung der einzelnen Organe nicht im gleichen Masze zurück. Im Gegenteil: Bei einigen Organen, nämlich am deutlichsten bei der Epiphyse, bei der Hypophyse, beim Auge, aber auch bei der Schilddrüse, ist die Entwicklung gefördert, während sie beim Pankreas dem Alter entspricht. Es sind also zum gröszten Teil gerade die für die Entwicklung wichtigsten Organe, die Drüsen mit innerer Sekretion, die am wenigsten durch die erzwungene Wachstumshemmung in Mitleidenschaft gezogen werden, deren Entwicklung sogar durch die erhöhte Sauerstoffzufuhr begünstigt wird. Das sich dies bei den verschiedenen Drüsen an verschiedenen Tagen äuszert, hängt wohl mit dem zeitlich verschiedenen Hervortreten der einzelnen Drüsen in der Gesamtheit des innersekretorischen Apparates zusammen. Auch andere lebenswichtige Organe, wie Leber und Niere, zeigen wenigstens keine nachweisbare Hemmung der Entwicklung, sondern nur der Grösze, die aber nicht in Miszverhältnis zur Körpergrösze steht. Am stärksten sind noch die Organe betroffen, die nicht lebensnotwendig sind, wie die Extremitäten und die noch nicht gebrauchte Lunge, eben so wie auch anscheinend das Gehirn. Dasz auch beim Auge eine relativ fortgeschrittene Entwicklung gefunden wurde, hängt vielleicht mit der Merkwürdigkeit zusammen, dasz die Augenentwicklung beim Hühnerembryo überhaupt in einem Miszverhältnis zu der Wichtigkeit dieses Organes während der Embryonalentwicklung steht.

In der Entwicklung der Organe aus verschiedenen Keimblättern konnten keine gesetzmäszigen Differenzen gefunden werden, ebenso keine Verschiebungen einzelner Teile innerhalb eines Organes. So geht z.B. die Mesenchymeinwanderung, die Linsen- und die Retinabildung im Auge parallel. Organe, die aus zahlreichen Einzelteilen, die unter einander gleich sind, zusammengesetzt sind, wie Leber und Niere, sind nur im Ganzen kleiner.

Betrachtet man die Entwicklung der Embryonen nicht nach dem Alter, sondern nach ihrem Gewicht, so ergibt sich ausnahmslos, dasz sie weiter vorgeschritten ist, als dem Gewicht entsprechen würde. Diese Differenz kann sehr weit gehen. Nach der Entwicklung des Auges berechnet müssten z.B. die Tiere 41, 45 und 50 über doppelt so schwer sein, als sie in Wirklichkeit sind.

Die Befunde an Hühnerembryonen stimmen ziemlich gut mit der in der Einleitung gegebenen Erwartung überein. Wahrscheinlich unterdrückt das höhere Sauerstoffangebot die vielleicht auf einem *relativen* Sauerstoffmangel beruhende Glykolyse im Ei. Hierdurch wird das Wachstum gehemmt. Die Differenzierung wird aber nicht in negativem Sinne beeinflusst, mindestens nicht in entsprechendem Masze, zum Teil sogar in progressivem Sinne. Erst bei sehr starker Hemmung des Wachstums leidet auch die

Entwicklung der meisten Organe. Wachstum und Entwicklung sind an sich mit einander verknüpft. Man kann die Zellteilung einmal rein von physikalischen Gesichtspunkten aus als ein Massenproblem ansehen: Die einzelne Zelle nimmt an Masse zu. Ein Miszverhältnis von Oberfläche zu Zellinhalt zwingt zur Zellteilung, die in diesem Falle lediglich der Möglichkeit einer Massenzunahme eines Organismus dient. Wenn (z.B. durch Unterdrückung der Glykolyse) die Massenzunahme der einzelnen Zelle gehemmt ist, wird seltener der Zwang zur Zellteilung bestehen und damit die Möglichkeit der differenten Potenzverteilung auf verschiedene Zellkomplexe. In diesem Falle würde die Verhinderung des Massenwachstums, die sich in einer verminderten Gewichtszunahme des Körpers zeigen würde, durch Verminderung der Zellteilungen zu einer Hemmung der Gesamtentwicklung führen.

Auf diese Weise liesze sich — unter bewuszter Vernachlässigung anderer Regulationsmechanismen, die sicher in dem ganzen Komplex mitspielen — erklären, warum Organe nach Erreichung bestimmter Gröszen ihr Wachstum einstellen. Ist durch Entwicklung eines genügenden Gefäszsystems für ausreichende Sauerstoffzufuhr gesorgt oder ist der Atmungsapparat der einzelnen Zelle für die betreffende Zellart weit genug entwickelt, so hört das Wachstum auf. Das gleiche gilt im Gegensatz zum Tumorgewebe für das Granulationsgewebe. Bei ihm liegt eine äuszerlich bedingte Atemschädigung durch Zerstörung des Gefäszsystems vor. Sobald diese durch Gefäszbildung beseitigt ist, hört das Wachstum auf. Die Narbe organisiert sich. Bei der Tumorzelle ist die Energiestoffwechselabweichung in der Zelle selbst begründet. Sie läszt sich nach den bisher gemachten Erfahrungen nicht durch erhöhte Sauerstoffzufuhr beseitigen. Unter erhöhten Sauerstoffpartialdruck gebrachte Mäusecarzinomzellen in vitro behalten ihre Malignität (geprüft an der Weiterverimpfbarkeit) und ihren abwegigen Stoffwechseltypus oder sie gehen zu Grunde. (Eigene unveröffentlichte Versuche.) Erhöhte Sauerstoffzufuhr kann nur bei Zellen mit intakter Atmung, aber nicht bei der Tumorzelle zu einer verstärkten Atmung, zur Unterdrückung der Glykolyse, des Wachstums, zur Tendenz, sich zu differenzieren, sich dem Bauplan des Gesamtorganismus einzuordnen führen, wohl aber — wie sich gezeigt hat — schneller zum Tod der Tumor*zelle* als der gesunden Zellen. Bei gesunden wachsenden Zellen kann die Differenzierung im Gegensatz zum Wachstum durch Sauerstoffüberangebot mindestens *relativ* gefördert werden.

Zusammenfassung.

Durch Aufzucht in Sauerstoff-reicher Atmosphäre wird das Wachstum von Hühnerembryonen gehemmt, die Differenzierung aber relativ und z.T. auch absolut gefördert.

Anatomy. — *Cephalization and the boundary values of the brain- and body sizes in mammals.* By S. T. BOK. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of May 20, 1939.)

The researches by VAN ERP TAALMAN KIP showed that in a series of rodents of progressive body sizes the total number of nerve cells in the cerebral cortex as well as the number of dendrites per nerve cell increase according to fixed rules: the number of nerve cells in proportion to the 5/6th power of the body surface and the total length of the dendrites per neuron to the square root of the linear body measures. This increase in length of the dendrites per neuron takes place in two ways: from mouse to rat by a *congruent enlargement of the tree of dendrites*, consequently with increasing radius but e.g. with the same number of dendrite endings, from rat to cavia on the contrary by *increased ramification of the dendrites*, where the radius remains the same but e.g. the number of dendrite endings increases. (From cavia to rabbit a congruent enlargement was observed again.)

These two ways of increase of dendrites may without constraint be compared to *growth* and *division*, two processes which also during increase in size of various other parts of the body occur alternately. Logically there would be, by the side of this growth and division of the dendrites, yet a third way, viz. a *division of the neurons*, where the division, which in the second case is restricted to the dendrites, would penetrate through the cell body and the neurite (here is not meant a division of already differentiated neurons, but a more potential process, which would take place before the ultimate histological differentiation and would result in a double number of neurons). Indeed, such divisions of neurons must occur in the mammals, considering the fact that according to the same investigation by VAN ERP TAALMAN KIP the number of neurons in the human cerebral cortex is considerably larger (about 34 ×) than it would be if the cortical structure of man would differ from that of the rodents only in such a way as, according to the above-mentioned rules, corresponds with the proportion in body size. Between these rodents and man, consequently, 5 such general neuron divisions would have taken place ($2^5 = 32$).

Such a general neuron division would influence the brain weight much more strongly than the first two ways: these two merely enlarge the size of the dendrite trees and the tissue components belonging to them, by neuron division also the spaces (white substance!) occupied by the cell bodies and by the mostly long neurites would be doubled.

This consequence is in agreement with the researches of Dubois, who from comparison of body weight and forebrain weight (weight of the part of the brain in front of the medulla oblongata) concluded that the brain weight of the mammals increases in two ways: in the first place with the body size (viz. in proportion to the 5/9th power of the body weight) and secondly by bounds, each being a doubling of the forebrain weight. The first way corresponds with the increase in number, demonstrated by van Erp Taalman Kip, of the cortical neurons in proportion to the 5/9th power of the body weight and the second with the neuron division advocated here [1]).

Dubois also ascribed the duplications of the brain weight to general neuron divisions (likewise conceived in a developmental stage before the differentiation of the neurons). However, he considered these divisions independent of the body size: they would be the manifestation of a higher degree of neural organization, for which reason he called them cephalization steps. Indeed, animal species with more cephalization steps (with a higher degree of cephalization) are usually also placed higher in the system of the mammals by the systematists. According to the view I suggested, however, the neuron divisions would be one of three ways by which the nervous system follows an increasing body size, so that indeed they would be connected with the body size. As the result of their action they would place the neural organization on a higher level, so that the fact of correlation between cephalization and degree of organization remains unaltered.

That by the side of this there is also a correlation between body size and degree of cephalization, as is required by the conception given here, is apparent from fig. 1, where the cephalization exponent [2]) of a number of mammals has been plotted against the logarithm of the body weight. The smallest mammals have the lowest cephalization exponents (the points farthest to the left are at the same time the lowest); with increasing body size (more to the right in the graph) the minimal as well as the maximal and the average cephalization exponents grow regularly larger. That both magnitudes are correlated is also apparent from the correlation index, calculated for these points at 0.50.

---

[1]) This agreement is not so easily interpreted as it might appear, since the changes in the space occupied by one neuron are not taken into account and besides DUBOIS weighed a large part of the brain whereas VAN ERP TAALMAN KIP only studied the cerebral cortex, so that the results may not yet be compared in detail, although upon the whole the similarity is striking. Moreover, DUBOIS likewise found 5 duplications between mouse and man.

[2]) As cephalization exponent has been used here the exponent c from the formula $E = 2^c . k . P^{5/9}$. E representing the forebrain weight, P the body weight and k a proportional factor equal for all mammals. For closely related species, where E is proportional to $P^{5/9}$, c is consequently the same; if k is well chosen, c would always be an integer, provided all cephalization steps were pure duplications.

For the graphs the data concerning forebrain- and body weights have been used, contained in the table on p. 9 of the dissertation by BRUMMELKAMP, who obtained these data from DUBOIS, KOHLBRUGGE, LAPICQUE, LAWSON–LAWRY, OWEN and WEBER. To these data I added those of *Balaenoptera Sibbaldi*, recorded by DUBOIS.

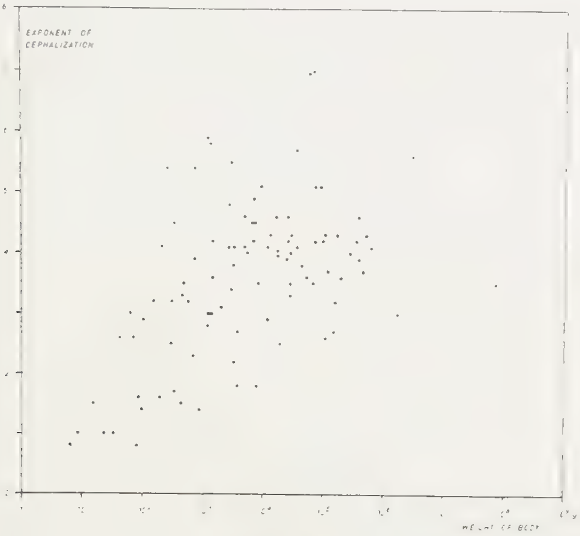


Fig. 1.

Consequently, given a connection between body size and cephalization, the question may be raised whether conditions may exist, in which the other ways of increase of dendrites have to give way to a cephalization step. This question may be illustrated by an example: to which general rule is it due that no rodents of human size are found but that with the increase in body size rather a number of neuron segmentations and consequently cephalization steps disturbed the continuous adaptation of the cerebral cortex to the increasing body size?

In fig. 2 the forebrain and body weights of the above-mentioned table have been logarithmically plotted against each other.

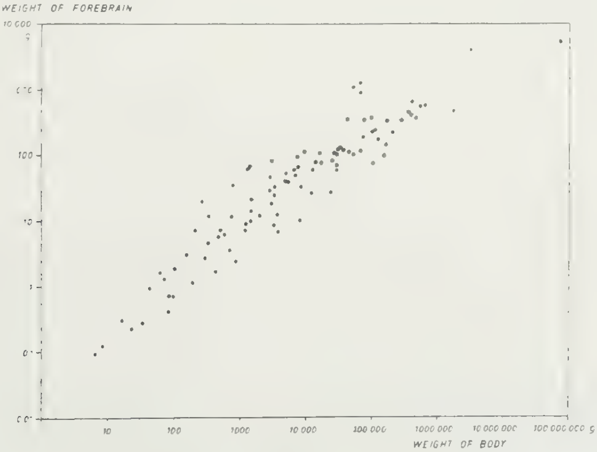


Fig. 2.

If the cephalization steps of DUBOIS would be clearly apparent from these numbers, the points in this graph would all be situated on (and not between) a set of parallel lines, which with the horizontal axis form an angle with a tangent = 5/9 and whose mutual vertical distance is = *log* 2. These lines have been drawn in figure 3: the points do not lie clearly upon them, many lie in between. That this is the case might partly be the result of the fact that the exact steps were found best by DUBOIS if fairly closely related species were compared. The larger groups linked to each other in this way by steps of 2 might have shifted somewhat irregularly, so that in this larger material they are no longer conspicuous. BRUMMELKAMP on the other hand thought that the steps in principle are twice as small as DUBOIS assumed and consequently amount to $\sqrt{2}$. Between every two successive lines in fig. 3 yet another line parallel to the preceding ones would have to be drawn and the smallness of these steps would promote the above-mentioned indistinctness. The question whether the cephalization steps amount to 2 × or to $\sqrt{2}$ × need not greatly influence the above-mentioned conception of the neuron divisions, since it is as yet unknown whether such a neuron division would immediately result in two neurons, each equally large as the neuron before division or first a certain percentage (e.g. $\sqrt{2}$ times) smaller, after which only in following species the original size would be reached by growth.

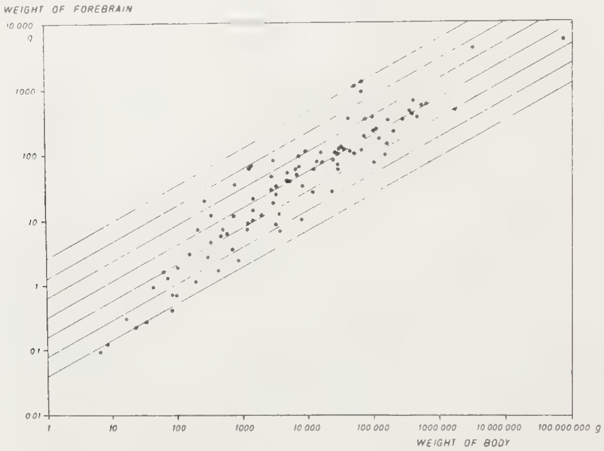


Fig. 3.

The relation points in fig. 2 form a connected field which has an exceedingly sharp boundary and a simple form. Unprejudiced consideration might lead to the conclusion that the relation between forebrain weight and body weight is a simple one, which is given by the straight axis of this field of points, and that among the various species represented here only relatively small and arbitrarily distributed deviations from this central relation occur. This relation itself would practically be a direct proportion between forebrain weight and body weight, the above-mentioned axis in this logarithmic graph forming an angle of nearly 45° (so with a tangent of nearly 1) with the horizontal axis.

If in this graph points, belonging to closely related animal species, are connected by lines, as has been done in fig. 4 for 1. a number of *Muridae*,

2. a number of *Siuridae*, 3. a number of *Canidae* and 4. a number of *Anthropoidae*, it appears that these lines indicating the "relation of the species" by no means run parallel to the above-mentioned axis: on an average they lie in the direction of a line, the angle of which with the horizontal axis has a tangent = 5/9, i.e. they demonstrate, as was to be expected, the connection found by DUBOIS between forebrain- and body weight of closely related species. Seen in the whole of the field of points, they start on the left (so with the animal species with the smallest body weight) on the left boundary of the field of points and run obliquely through the field until at the lower boundary they must of necessity end. If we should arrange the mammals 1^0 according to the species and 2^0 according to body size, we should not, without passing the boundaries of this field, be able to come gradually without steps from mouse to man. At least at the lower boundary of the field we should have to shift to a higher line of species and on this higher line we should not "start" before the left upper boundary of that field. These two boundaries consequently show in some way or other conditions which restrict the growth and division of the dendrites and where nothing but a neuron division (cephalization step) must occur (which of course by no means excludes that such a step may occur also in the middle of the field of points).

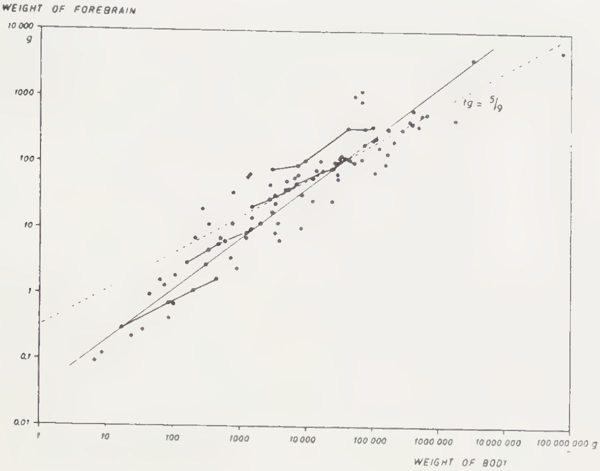


Fig. 4

These boundaries of the field are exceedingly sharply drawn and therefore it is the more surprising that there has never been paid attention to them. Fig. 5 shows that these boundaries form straight lines. Moreover, the directions of these straight lines point to simple relations between forebrain weight and body weight: the tangent of the gradient of the lower boundary line is 2/3, that of the upper one 4/3, i.e. along the lower boundary lie animal species, the forebrain weight of which is proportional to the 2/3 power of the body weight, along the upper boundary they are proportional to the 4/3 power. Leaving out of consideration the in this respect relatively small differences in detail in the forms of these animal species, we may for a proportion with the 2/3 power of the body weight

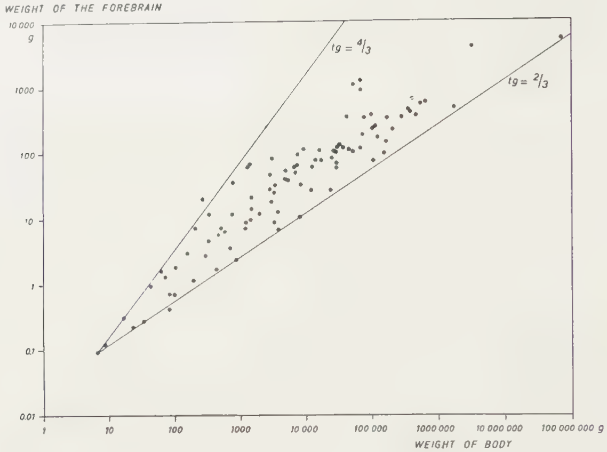


Fig. 5.

read a proportion with the body surface, so that the lower boundary is given by species, the forebrain weight of which is proportional to the body surface, and the upper boundary by species of which the forebrain weight is proportional to the square of the body surface. That means that the ratio *brain weight: body surface* is not lowered below a certain figure and that the ratio *brain weight: square of the body surface* does not rise above a certain figure. (These figures are those of the smallest animal, the *vesperugo pipistrella.*)

On the ground of these data the graphic representation may be more clearly arranged by plotting vertically not the brain weight itself but this ratio *forebrain weight: body surface*. In reality in fig. 6 the ratio *forebrain weight: 2/3 power of the body weight* was plotted vertically, not the body surfaces but the body weights being measured. Since it is further known that the forebrain weights, apart from a few exceptions, are proportional to the surface of the neocortex, after composing the field of points the zero of the vertical axis has been shifted in such a way that the figures placed there correspond to the ratio *neocortical surface: body surface* or to the neocortical surface per unit of body surface. For the same reason not the body weight but the body surface was plotted horizontally [1]).

Owing to this transformation the field of points appears to have assumed a surprisingly simple form, viz. that of an isosceles trapezium, the basal angles of which are 45° and the lower side is three times longer than the upper one.

---

[1]) Vertically, therefore, has been plotted $\log eE : pP^{2/3}$, in which E represents the forebrain weight, P the body weight and e and p are proportional factors, so that eE represents the exact neocortical surface for man and $pP^{2/3}$ his body surface. Horizontally $\log pP^{2/3}$ has been plotted.

In the lower side we recognize the lower boundary of the field in the figures 2 and 5. The fact that this lower boundary had to be horizontal here follows from the previously known peculiarity that of all species of mammals represented here the neocortical surface per unit of body surface is the same, viz. equal to the minimum value. The new figure (6) shows how accurately this minimum value has been maintained. At the same time it is striking that with so many animal species this minimum value occurs (apart from small differences, to which we can attach no value, owing to the relatively great uncertainty in the determinations of the body weight). In fig. 6 the lower boundary has been drawn through the relation point of the smallest of the represented species (*vesperugo pipistrella,* indicated by the characters V. P.). There the relation between the brain weight and the 2/3 power of the body weight is 0.01 × 2.8. The latter factor is in man 72 (over 25 times as large). In 14 of the 95 represented species this factor is at most only 20 % larger, and to these species belong animals with very different body sizes, as is apparent from the table of page 522.

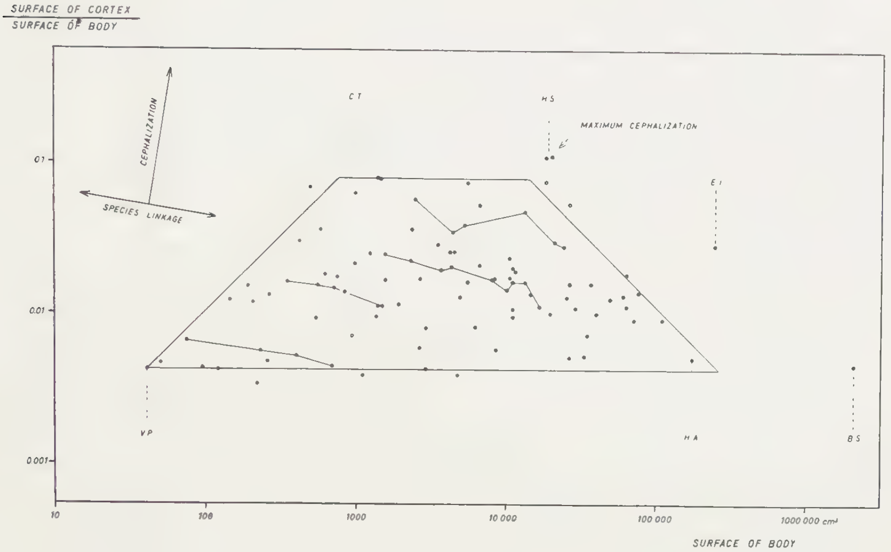


Fig. 6: The relation between neocortical surface and body surface of a number of mammals, compared with the body surface.

V.P. : *Vesperugo pipistrella*
C.T. : *Cercopithecus talapoin*
H.S. : *Homo sapiens*
H.A. : *Hippopotamus amphibius*
E.I. : *Elephas indicus*
B.S. : *Balaenoptera sieboldii*

Opposite this lower boundary lies a likewise horizontal upper boundary. Consequently, along this boundary too the neocortical surface per unit of body surface has a constant value and this means a maximum value (only man is represented above this maximum value; *pithecanthropus erectus*, as far as this may be determined, lies just within it). The maximum value

Species	Data obtained from	Body weight (in grammes)	Forebrain weight (in grammes)	Forebrain weight divided by the 2/3 power of the body weight (× 100)
Balaenoptera Sibbaldi (Balaenoptera sieboldii GRAY)	DUBOIS	74000000	5243	3.0
Hippopotamus amphibius (Hippopotamus amphibius L.)	WEBER	1755000	479	3.3
Sus scropha domesticus (Sus scropha domesticus L.)	LAWSON–LAWRY	104539	73.9	3.3
Manis Javanica (Manis Javanica L.)	KOHLBRUGGE	8000	10.0	2.5
Lagostomus trichodactylus (Lagostomus trichodactylus BENN.)	OWEN	3854	6.69	2.7
Erinaceus europaeus (Erinaceus europaeus L.)	WEBER	885	2.36	2.6
Mus norvegicus (Rattus norvegicus (ERXL.))	WEBER	448	1.69	2.9
Mus rattus (Rattus rattus (ERXL.))	DUBOIS	200	1.137	3.3
Scotophilus gigus (Scotophilus gigus DOBSON)	WEBER	101	0.666	3.1
Crocidura schweizeri (Crocidura schweizeri PETERS)	WEBER	82	0.409	2.2
Vespertilio murinus (Vespertilio murinus L.)	DUBOIS	33.3	0.283	2.7
Rinolophus ferrumequinum (Rhinolophus ferrum-equinum SCHREB.)	DUBOIS	23.5	0.230	2.8
Crocidura araneus (Crocidura araneus BLAS.)	LAPICQUE	8.8	0.128	3.0
Vesperugo pipistrella (Pipestrellus pipestrellus SCHREB.)	DUBOIS	6.5	0.096	2.8

appears to be 19 × the minimum value. *With the mammals the neocortical surface per unit of body surface is consequently limited between a minimum and a 19 times larger maximum value.* This limitation is not such that with one species the highest and with another the smallest value occurs, but a conspicuously large number of species of very different body sizes lie exactly on these boundary values.

The right side of the trapezium points to a maximum in the absolute size of the neocortical surface: to all points of this line, namely, belongs the same neocortical surface (and the same brain weight). This follows from the consideration that along this line (owing to its gradient of —45°) with an increase of the horizontally plotted magnitude (log S or log surface of the body) the vertically plotted magnitude (log N/S) decreases by the same figure, so that their sum remains the same: log S + log N/S consequently is constant, so that log S + log N —log S or log N and therefore also N along this line is constant. Points on the left of this line point to a smaller neocortical surface, points on the right to a larger one. However, practically no points are found on the right of this line; only two points (the elephant and the whale) form an exception.

Consequently, with the species of mammals represented here the neocortical surface remains below a certain maximum value. This maximal value of the neocortical surface (and consequently also of the forebrain weight) is, for example, specific to *hippopotamus amphibius* (H. A. in fig. 6).

The neocortical surface (and the forebrain weight) is minimal with the species represented most to the left below, *vesperugo pipistrella*, which minimum according to the figure is 19^3 times smaller than the maximum. The neocortical surface (and the forebrain weight) consequently varies 19^3 times. This variation, therefore, is equally large as the variation of the body surface.

In the left boundary we recognize the left upper boundary of the field of points in the figures 2 and 5. Along this boundary the relation *neocortical surface: square of the body surface* (or *brain weight: body surface*^{2}) is constant and maximal. This follows from the consideration that along this line, owing to its gradient of 45°, the vertically plotted value rises as much as the horizontally plotted one, so that their difference is constant: log N/S — log S = constant, so that log N — log S — log S or log N — 2 log S or log N/S^2 and consequently also N/S^2 is constant.

Consequently also the relation between the neocortical surface (or the forebrain weight) and the square of the body surface with the mammals is limited by a certain maximum value. The biological significance of this limitation is not so selfevident as that of the three above-mentioned limitations. Possibly this relation has a functional significance, in connection with the integrating function of the nervous system. The first two limitations point to an intimate relation between the size of the nervous system and the body *surface* (by which is meant a two-dimensional body measure, which by no means need to be e.g. the surface of the skin). If now this body surface (this two-dimensional body measure) determines the number of afferent (and efferent) stimulation tracts, then a body surface twice as large would double this number of stimulation tracts and the receiving and transmitting apparatuses linked to them. At the same time, however, it must be possible to connect each of these tracts with twice that number of other tracts, so that the number of possible connections — expressed very schematically — maximally would become 2^2 times larger. It is possible that this consideration indicates the direction of the significance of this fourth boundary, which determines the maximum of the relation between neocortical surface (or forebrain weight) and the square of the body surface. The fact that this maximum is indeed determined, may with certainty be read from the many points lying along this left side of the trapezium.

The maximum variation of this relation N/S^2 is, like that of the neocortical surface and of the body surface, 19^3 times. The minimum value is situated near the lower right angle of the trapezium (the hippopotamus).

Recapitulating we may consequently state that the brain- and body measures in the present species of mammals are subject to the following four restrictions:

1. the relation *neocortical surface* : *body surface* (or the quantity of neocortical surface per unit of body surface) with the mammals does not descend below a certain minimum value (present e.g. in *vesperugo pipistrella*, see V.P. in fig. 6);
2. the same relation does not exceed a certain maximum value, which is 19 × larger than the mentioned minimum value;
3. the relation *neocortical surface* : *square of the body surface* does not exceed a special value (likewise present in *vesperugo pipistrella*);
4. the neocortical surface does not exceed a certain maximum value (19^3 times the minimum value, present in *vesperugo pipistrella*).

It appears from the regularity of the trapezium that between these four restrictions there must exist a mutual relation.

To these four restrictions only very few exceptions occur. The relation points for man are found just outside the upper right angle of the trapezium, so that it is doubtful whether man forms an exception. Only the elephant and the whale (E.I. and B.S. in fig. 6) lie definitely outside. However, the great majority of the represented species lie inside and so many points just touch the described boundary lines that these boundaries undoubtedly must have an intrinsic value, as regards the structure of the nervous system as well as the restriction of the idea mammal.

Which exactly is this significance cannot yet be ascertained. It will be of great value to make the same determinations on other classes in order to find out whether other classes may be limited in a similar way and of what nature the differences between those classes, considered in this way, will then appear to be.

Similarly it will be of great value to make a chart in this way of the extinct species of mammals, in order to see whether and how the trapezium changes with time. Possibly this will also throw light upon the few, still living, very large species which lie outside the trapezium.

It is interesting to see how the relation of the species and the cephalization are expressed in this trapezium graph.

In a simple way we can conclude that the ideal line of the relation of the species, which in fig. 5 had a gradient with a tangent $= 5/9$, here is a descending line with a tangent $= -1/6$. This is also demonstrated by the fact that the same points which are mutually connected in fig. 5 are likewise connected in fig. 6. The cephalization steps, consequently, lie here in a steep direction, which is characterized by a gradient with a tangent $= 5/6$ (see the arrows drawn on the left above). The upper angular point on the right consequently has the highest degree of cephalization. Here *pithecanthropus erectus* is found, and one cephalization step outside man (man and woman indicated separately). Along the lines of the relation of the species, so following the increasing body size with growth or division of the dendrites, the relation *neocortical surface* : *body surface* descends constantly. This is clear, owing to the fact that the brain weight and consequently the neocortical surface increase in proportion to the 5/9 power of the body weight or in proportion to the 5/6 power of the body surface, i.e. somewhat less rapidly than the body surface. If our starting-point lay not so high in the figure (if we had started from a not specially highly cephalized species), this descent would come to a stop when the relation *neocortical surface* : *body surface* reaches the minimum value (lower side of the trapezium). With a then consequently necessary enlargement by neuron division we may not start on the following line of species before N/S^2 has the maximum value. If our starting-point lay higher (or if we reach such a higher point after a general neuron division), then the limit of the lines of the relation of the species, and consequently the necessary indication for a neuron division, is reached when the neocortical surface reaches its maximal size.

The observed fact, that the indicated simple limits, fixed for the mammals to the relation between brain- and body size, form necessary indications for the cephalization steps (general neuron divisions) is a second confirmation of the view that the cephalization steps are connected with the increasing body sizes.

Medicine. — *On the Time and Place of Experimental Myelin Degeneration in the Optic Pathways.* By R. G. MEADER [1]), Yale University, New Haven. (From the Neurological Laboratory of the University of Amsterdam. Director: Prof. Dr. B. BROUWER). (Communicated by Prof. B. BROUWER.)

(Communicated at the meeting of May 20, 1939.)

The time at which secondary degeneration of the medullated nerve fibers is first visible has been studied chiefly in the peripheral nervous system. Systematic examinations concerning this problem in the central nervous system are rare. SPIELMEYER (1929) summarized the knowledge in both fields. To that have been added more recently the observations of PARKER and PAINE (1934) on peripheral nerves and the extensive review of ROSSI and GASTALDI (1935) on both central and peripheral nerve fibers. The general concensus of recent opinion seems to be that in the fibers of the central nervous system as well as in those of the peripheral system, degeneration of the axis cylinder and of the myelin sheath in the distal stump begins at the lesion and proceeds toward the periphery of the nerve fiber. It seems worth while to make available the observations made on the time and place sequence of degeneration in the medullary sheaths of the optic pathways following enucleation of an eye.

Five rabbits were sacrificed at 96, 133, 180½, 232 and 279 hours, respectively, after enucleation of the left eye by Prof. Dr. W. P. C. ZEEMAN. The central nervous system was prepared by the Marchi method in the usual way. Projection drawings were made of the optic nerves, chiasma, tracts, corpora geniculata externa, and corpora quadrigemina antica. Details of granule distribution were filled in from microscopical observations. (See figs. 1, 2, 3).

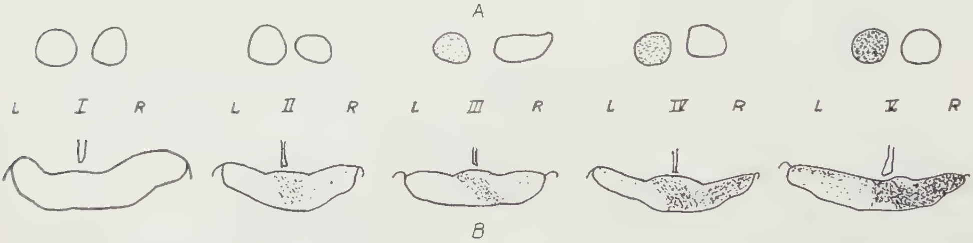


Fig. 1. Transverse sections through (*A*) the optic nerves and through (*B*) the caudal part of the optic chiasma of the rabbit, showing the amount of Marchi degeneration at successive periods after enucleation of the left eye. I. 96 hours, II. 133 hours, III. 180½ hours, IV. 232 hours, V. 279 hours.

---

[1]) Fellow of the ROCKEFELLER Foundation 1938—'39.

The general results may be summarized as follows: Marchi degeneration granules appear earliest (96 hours) in the more distal parts of the optic nerve fibers whose cell bodies have been removed. Between 133 and 180½ hours the portions of the nerve fibers nearest the lesion begin to show indubitable degeneration and the process reaches its height throughout the length of the fiber only after 279 hours.

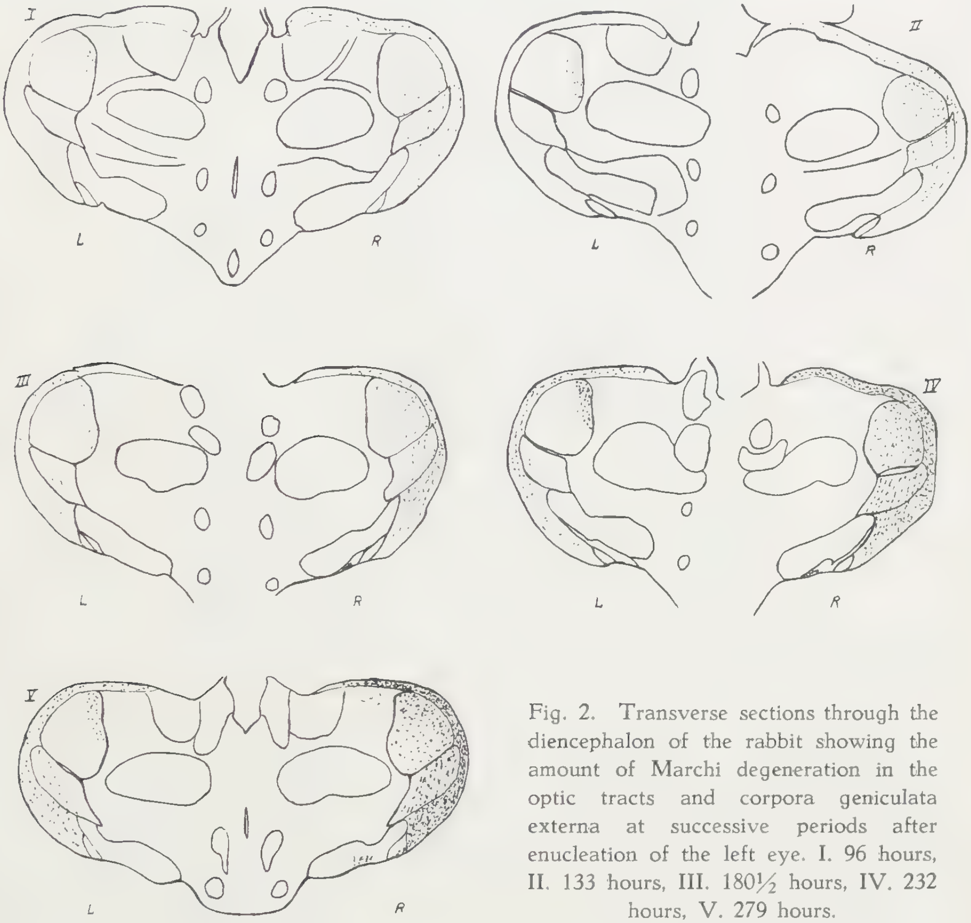


Fig. 2. Transverse sections through the diencephalon of the rabbit showing the amount of Marchi degeneration in the optic tracts and corpora geniculata externa at successive periods after enucleation of the left eye. I. 96 hours, II. 133 hours, III. 180½ hours, IV. 232 hours, V. 279 hours.

The detailed observations on each experimental animal are presented below. The number in brackets is that assigned to the series in the neurological laboratory of the University of Amsterdam.

96 hours after enucleation (B 621).

Degeneration granules are not to be seen in the optic nerve and chiasma, nor in the ipsilateral (left) tract. A few black grains in rows appear in the crossed (right) tract as it approaches and embraces the corpus geniculatum externum. In the dorsomedial nucleus of this latter body many fine

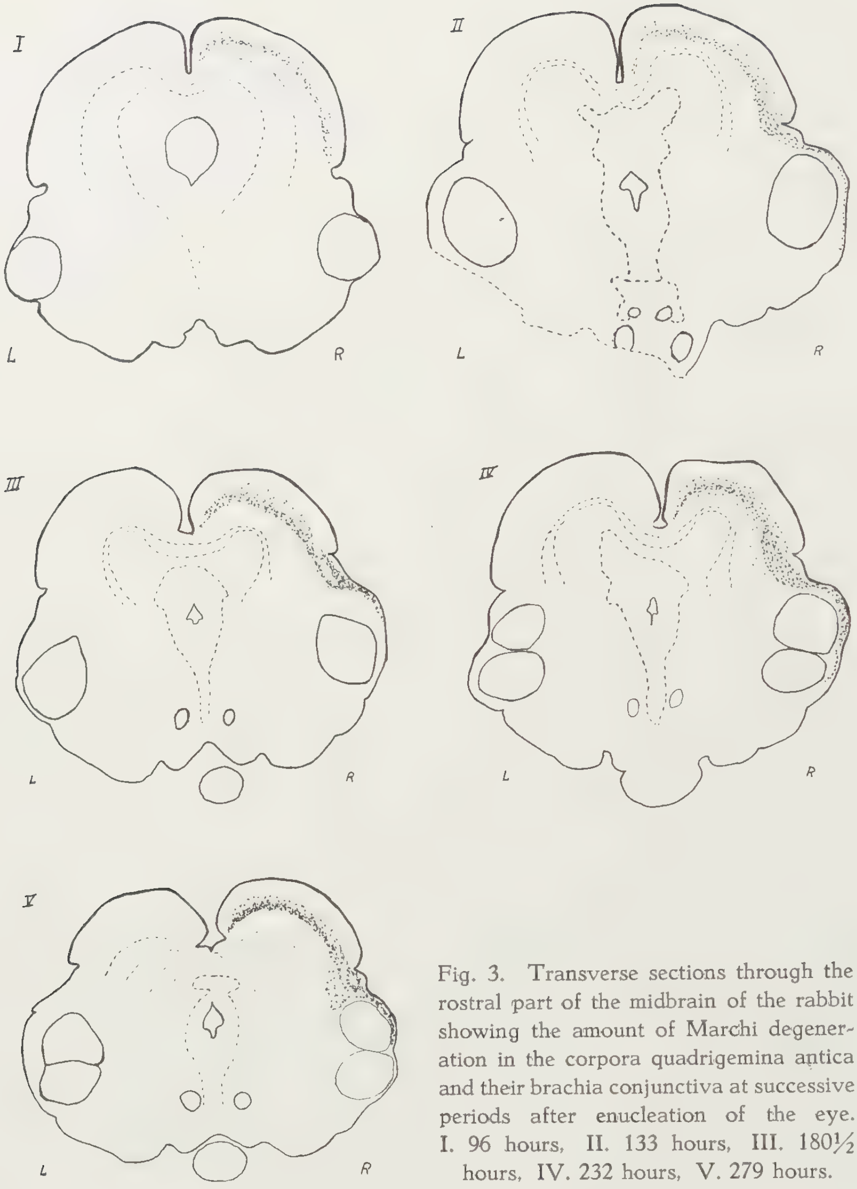


Fig. 3. Transverse sections through the rostral part of the midbrain of the rabbit showing the amount of Marchi degeneration in the corpora quadrigemina antica and their brachia conjunctiva at successive periods after enucleation of the eye. I. 96 hours, II. 133 hours, III. 180½ hours, IV. 232 hours, V. 279 hours.

granules are visible, scattered all through it on the right but confined to its medialmost region on the left side. The ventrolateral nucleus of the geniculate body is clear on the left and the few grains that appear on the right are situated in the fiber bundles traversing it. Many larger black grains are evident in the brachium conjunctivum and in the superficial fiber layer of the right corpus quadrigeminum anticum (figs. 3, 4). Smaller granules lie in the portion of the superficial cell stratum adjacent to the fiber stratum. The left optic tectum and its brachium show no degeneration granules.

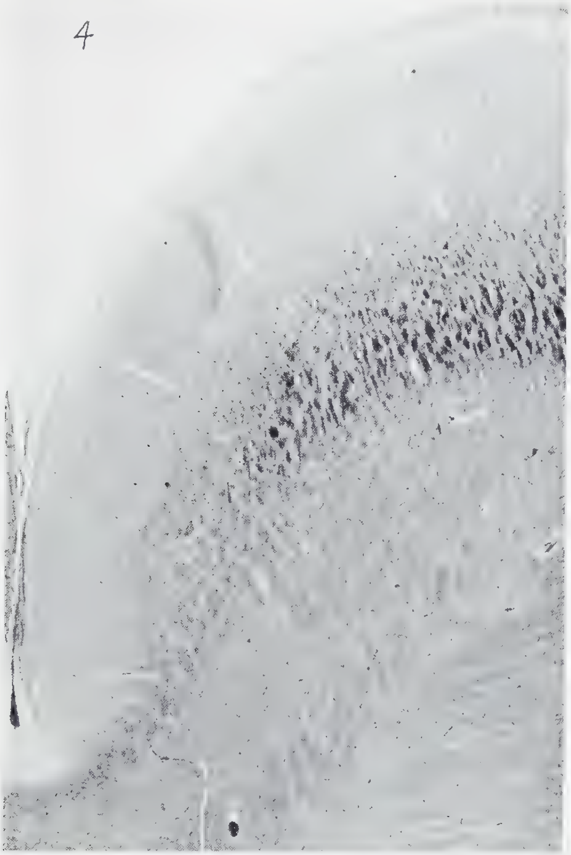
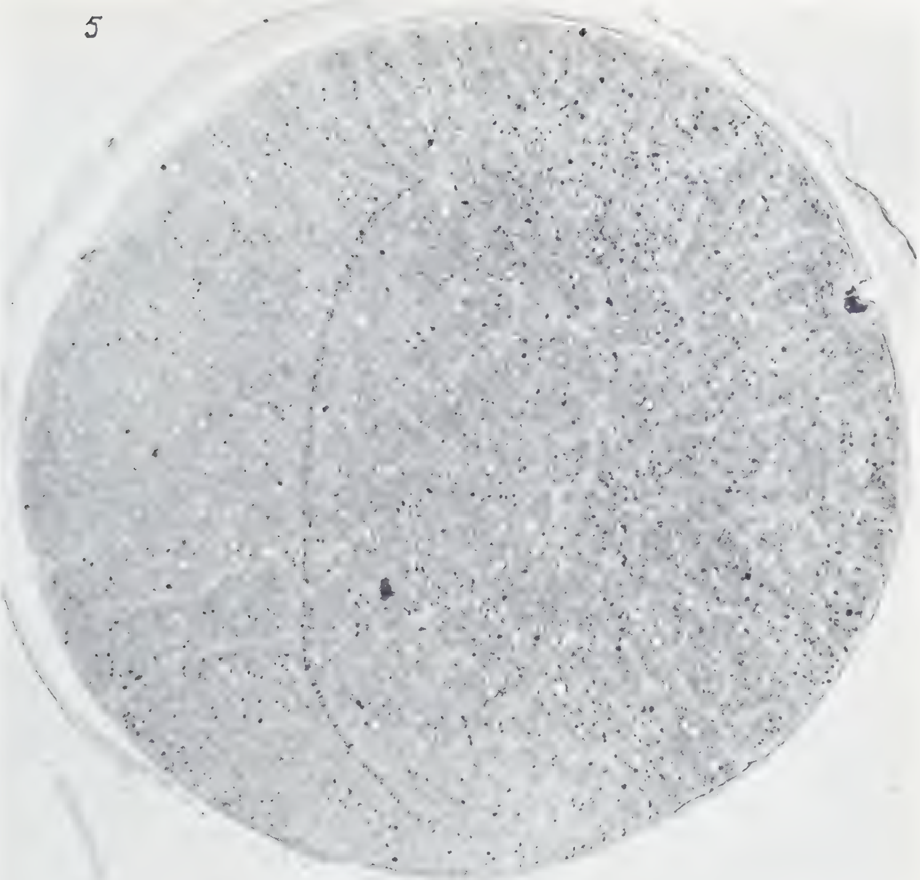


Fig. 4. Photomicrograph of the medial part of the right corpus quadrigeminum anticum 96 hours after enucleation of the left eye. Cross-section. Marchi stain.

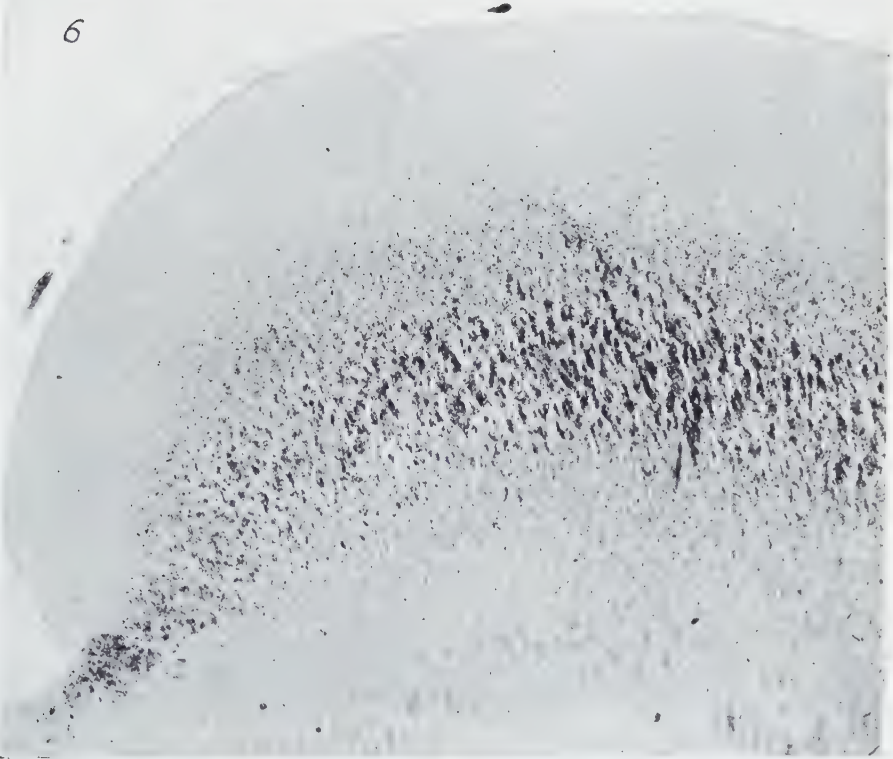
Fig. 5. Photomicrograph of the left optic nerve 180½ hours after enucleation of the left eye. Cross-section. Marchi stain.

Fig. 6. Photomicrograph of the right corpus quadrigeminum anticum 180½ hours after enucleation of the left eye. Cross-section. Marchi stain.

5



6



133 hours after enucleation (B 616).

The optic nerves are free from degeneration products but in the chiasma a few granules are evident among the interlacing bundles. Increasing numbers of larger grains appear in the right tract as it reaches the geniculate body. They tend to lie parallel to the course of the fibers. There is no clear degeneration in the uncrossed optic tract. The condition in the right and left lateral geniculate bodies is merely an accentuation of that seen after 96 hours. On the right side, the granules seem to be a bit more numerous in the lateral portion of the dorsomedial nucleus than in its medial portion, due in part to the larger granules present in the bundles of optic fibers that traverse that nucleus. The ventrolateral nucleus on the right shows similar but very few groups of granules in the fibers passing through it. There are none on the left. In the right corpus quadrigeminum anticum the number of large and small granules in the brachium conjunctivum and in the superficial fiber and cell strata has increased. Those in the cellular layer are chiefly of the small variety. The left colliculus and its brachium are not degenerated.

180½ hours after enucleation (B 617).

Large and small black grains are sharply distinctive of the left optic nerve (fig. 5) and are absent from the right. They are not numerous on the left, however, and most of the optic fibers appear to be still intact. In the chiasma, the grains form obliquely oriented rows in the region of the interlacing bundles. They are distributed throughout the right optic tract in short rows parallel to the normal appearing fibers. They become more numerous in the vicinity of the corpus geniculatum externum. The left tract contains granules which become larger and more numerous as it approaches the optic centers but they are never very prominent. In the corpora geniculata externa and in the right corpus quadrigeminum anticum (fig. 6) the only change from the earlier picture is the continued increase in the number of degenerating granules.

The fasciculus accessorius optici anterior (BOCHENEK) and the tractus peduncularis transversus on the side opposite the enucleated eye exhibit degeneration.

232 hours after enucleation (B 618).

The only change from the preceding series is the intensification of the degenerative process, that in the primary optic centers continuing to be more marked than that in the optic nerve. Degeneration in the left optic tract (uncrossed fibers) is clearly seen, being concentrated chiefly in the dorsal part of the tract.

279 hours after enucleation (B 620).

The myelin degeneration is not yet at its height but the entire optic pathway of the first neurone is involved. The left optic nerve, the crossing fibers of the chiasma, and the right optic tract, lateral geniculate body and anterior colliculus are all loaded with large and small black granules. The right fasciculus accessorius optici anterior (BOCHENEK) and tractus peduncularis transversus are affected in a similar way. The degenerated fibers in the ipsilateral (left) optic tract are few and lie chiefly in the dorsal part of the tract. Those in the left corpus geniculatum externum are confined to the medial part of its dorsomedial nucleus except for the fibers of passage in the ventrolateral nucleus. There is no degeneration in the left corpus quadrigeminum anticum.

Discussion.

It is clear that within the optic pathways of the first neurone in the rabbit, degeneration of the medullary sheaths can be recognized in Marchi preparations 4 days after enucleation of the eye, but only at the distal end of the fiber, i.e., in or near the terminal centers. Degeneration in the proximal part of the fiber, i.e., in the optic nerve, can be recognized 7½ days after operation. It is to be noted, also, that degeneration of the fibers in the uncrossed optic tract is not clearly recognizable until nearly 10 days have passed, considerably later than in the crossed tract (5½ days).

The first mentioned results are somewhat surprising in the light of the studies made on the time and place of appearance of degeneration in the medullary sheaths of severed peripheral nerves. It was stated by HOWELL and HUBER (1892) and confirmed by MÖNCKEBERG and BETHE (1799) and ROSSI (1912) that the first evidences of myelin segmentation appear on the 4th day after operation. These observations, however, were made on the proximal part of the distal stump near the lesion. Although I found Marchi granules in the optic fibers 4 days after enucleation, they were situated, not at the site of the lesion, but near the other end of the fiber. Since the preparations do not contain the portion of the optic nerve immediately next to the lesion, nothing can be said on that point. If Marchi granules were present, however, they must have been restricted to the first few millimeters of the transected nerve that were not removed with the brain. Only after 7½ days were any granules found in the most rostral sections close to the region of the lesion, a finding in partial agreement with SPIELMEYER's statement (1929) that Marchi degeneration begins about 8 days postoperatively. SPIELMEYER made no distinction, however, between the myelin degeneration of the proximal and distal parts of the distal stump. The only clearcut study of the direction and rate of myelin degeneration was made in the peripheral (lateral line) nerve of a fish (Ameiurus) by PARKER and PAINE (1934). Although their report indicates that the distal stump of a severed nerve undergoes myelin degeneration from the lesion toward the periphery at a rate of approximately 3.1 cm per day at a temperature of 18° C, it is hardly comparable with these observations on the central nervous system of a mammal. The disagreement with regard to the direction of degeneration may be due to the difference between central and peripheral nerve fibers or between mammals and fishes. So far as I have been able to find out, there have been no similar studies made in mammals. ROSSI and GASTALDI (1935) studied the degeneration and regeneration of the optic nerve in rabbits, but they used only silver preparations, in order to follow the changes in the axis cylinders. SPIELMEYER (1929) mentioned in a footnote a case of optic nerve disease in man in which the nerve itself was free from degeneration products while the tract still contained a rich supply of them. He interpreted the condition as one in which the degeneration products had been already resorbed from the nerve because of its better circulatory relations due to its many septa, whereas the more compact tract still retained them. He gave no data concerning the duration of the injury or the condition of the optic centers. In view of the observations reported here, it seems possible that the nerve showed no degeneration products because they had not yet been formed there.

In the rabbits investigated, the full Marchi picture could not be obtained until later than 279 hours although there were great numbers of degenerated fibers present throughout the entire optic pathway of the first neurone as early as 233 hours. This observation is in approximate agreement with SPIELMEYER (1929) who considered the high point of the Marchi stage to be about the 12th day. ROSSI and GASTALDI (1935) found a gradual increase in the formation of myelin ellipsoids in the optic nerve of the rabbit during the 4th to the 10th days, after which time that phase of the process was completed. In order to secure the full degeneration, it seems safest to allow at least 12 days to elapse after operation.

The gradual increase in the intensity of the Marchi staining depends in part upon the increase in the number of fibers undergoing degeneration. According to MÖNCKEBERG and BETHE (1899), the finer medullated fibers of a peripheral nerve degenerate more slowly than the coarser ones. They believed, also, that sensory fibers break up faster than do motor fibers. My preparations confirm the increase with time in the number of Marchi granules at any given level of the optic tract but it is not clear whether the larger fibers are the first to go or not. It might be argued that since the granules appear earliest in the terminal centers where the fibers are finely medullated, it is the finer fibers which disintegrate first. That is only part of the story, however, for in the colliculus anterior at the earliest stage are found larger myelin ellipsoids, also, supposedly from thicker medullary sheaths. It seems likely, therefore, that the degeneration here is more a factor of distance from the proximal part of the fiber than of fiber size, for both large and small fibers are involved.

The earlier recognition of degeneration in the crossed fibers of the optic tract than in the uncrossed fibers may be due to one or both of two factors.

There may be a true differential in the time of their reaction to separation from their cell bodies which might rest upon a difference in the size of the crossed and uncrossed fibers, or upon some other unrecognized difference. It may, also, be that degeneration occurs just as soon in comparable fibers of the two sides but that the uncrossed fibers are so few in number and so diffused among the normal crossed fibers from the uninjured side that they are not recognizable and are considered artefacts until a sufficient number of granules have formed to make their relationships clear.

Summary.

1. Marchi degeneration granules are visible in the contralateral anterior colliculus, in both external geniculate bodies and in the distal parts of the contralateral optic tract 96 hours after the enucleation of one eye in the rabbit. They are clearly recognizable in the severed optic nerve 180½ hours after the operation. The full Marchi picture appears only after more than 279 hours.
2. The degenerative process takes place at an unequal rate in the fibers of the optic pathway. It is recognizable earlier in the crossed optic tract than in the uncrossed bundle. In the crossed tract, moreover, there is an increase in the number of Marchi granules at any one level proportional (within limits) to the postoperative survival of the animal.

LITERATURE CITED.

HOWELL, W. H., and G. C. HUBER, A physiological, histological and clinical study of the degeneration and regeneration in peripheral nerve fibers after severance of their connections with the nerve centers. J. Physiol., Lond., **13**, 335—406 (1892).

MÖNCKEBERG, G., and A. BETHE, Die Degeneration der markhaltigen Nervenfasern der Wirbelthiere unter hauptsächlicher Berücksichtigung des Verhaltens der Primitivfibrillen. Arch. mik. Anat., **54**, 135—183 (1899).

PARKER, G. H., and VIRGINIA L. PAINE, Progressive nerve degeneration and its rate in the lateral-line nerve of the catfish. Am. J. Anat., **54**, 1—25 (1934).

ROSSI, O., Regenerative Vorgänge im Nervus Opticus. J. Psychol. Neurol., Leipzig, **19**, 160—186 (1912).

ROSSI, O., and G. GASTALDI, La rigenerazione del tessuto nervoso nei vertebrati superiori. Rivista critica con dati personali. Rivist. patol. nerv. ment., **46**, 1—366 (1935).

SPIELMEYER, W., Degeneration und Regeneration am peripherischen Nerven. A. BETHE, G. VON BERGMANN, G. EMBDEN and A. ELLINGER. Handb. norm. path. Physiologie, **9**, 285—338 (1929).

BERICHTIGUNG

zur Mitteilung von W. P. POSTMA: *Einige Bemerkungen über den Einfluss der Nitratreduktion auf die Atmung der Wurzeln*

in Proceedings Vol. XLII, N⁰. 2, 1939.

A. Tabelle III (Seite 184) soll heissen:

1. Reihe: Avena-Pflanzen, gezüchtet im Licht auf KNOP-Lösung ohne N. Analysiert nach 3, 7 und 11 Tagen.

2. Reihe: Avena-Pflanzen 3 Tage gezüchtet auf KNOP-Lösung ohne N, dann weiter auf KNOP-Lösung. Analysiert nach 7 und 11 Tagen.

Die Zahlen geben mg N in je 100 Pflanzen an.

	KNOP ohne N			KNOP		
Analysiert nach	Lösl. N	Eiweiss-N	Gesamt-N	Lösl. N	Eiweiss-N	Gesamt-N
3 Tagen	18.8	30.4	49.2			
7 Tagen	12.9	26.2	39.1	13.2	44.3	57.5
11 Tagen	13.4	19.1	32.5	7.1	56.4	62.5

B. Seite 185, Z. 9—14 ist folgendermassen zu lesen:

Pflanzen auf destill. Wasser je Stunde		0.029	m.mol. CO_2
„ „ —N —Glukose „ „	im Mitt.	0.031	„
„ „ +N —Glukose „ „	im Mitt.	0.040	„
„ „ —N +Glukose „ „		0.059	„
„ „ +N +Glukose „ „		0.360	„